Numerical approaches for integrable equations in higher dimensions

C. Klein, IMB, Dijon

Outline

- Introduction
- Fourier spectral methods
- Multidomain spectral methods
- Time integration, stiff integrators

Hopf equation

- hyperbolic conservation law for $u(x,t)$, initial data $u_0(x)$

  $u_t + 6uu_x = 0$, $u(x,0) = u_0(x)$

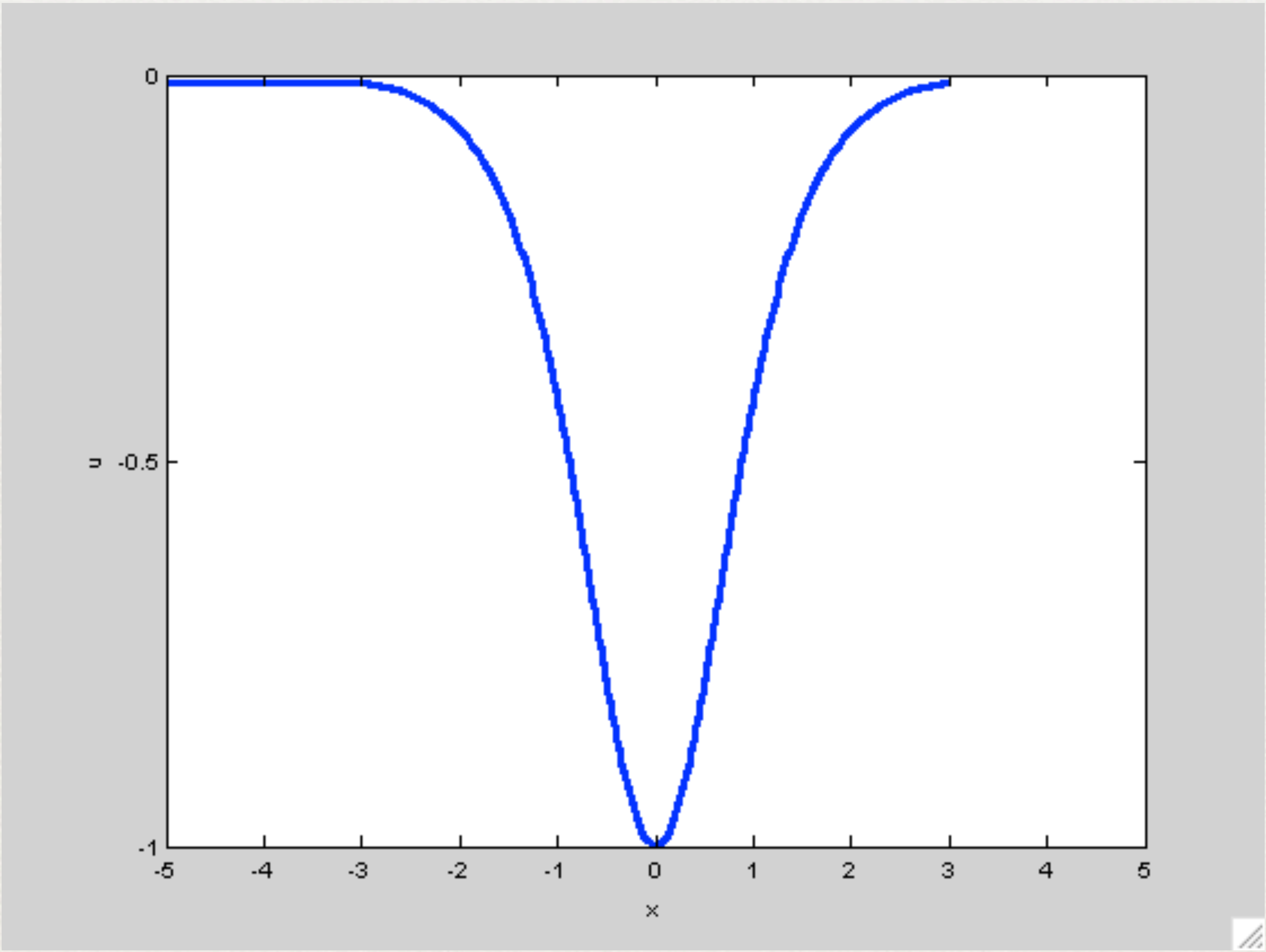
- solution with the method of characteristics

  $u(x,t) = u_0(\xi)$, $x = 6tu_0(\xi) + \xi$

- critical time $t_c = \dfrac{1}{\min_{\xi\in\mathbb{R}}\left[-6u_0'(\xi)\right]}$, gradient catastrophe,
  $t > t_c$: solution multivalued (shock)

Example: $u_0 = -\text{sech}^2 x$

Example: $u_0 = -\mathrm{sech}^2 x$



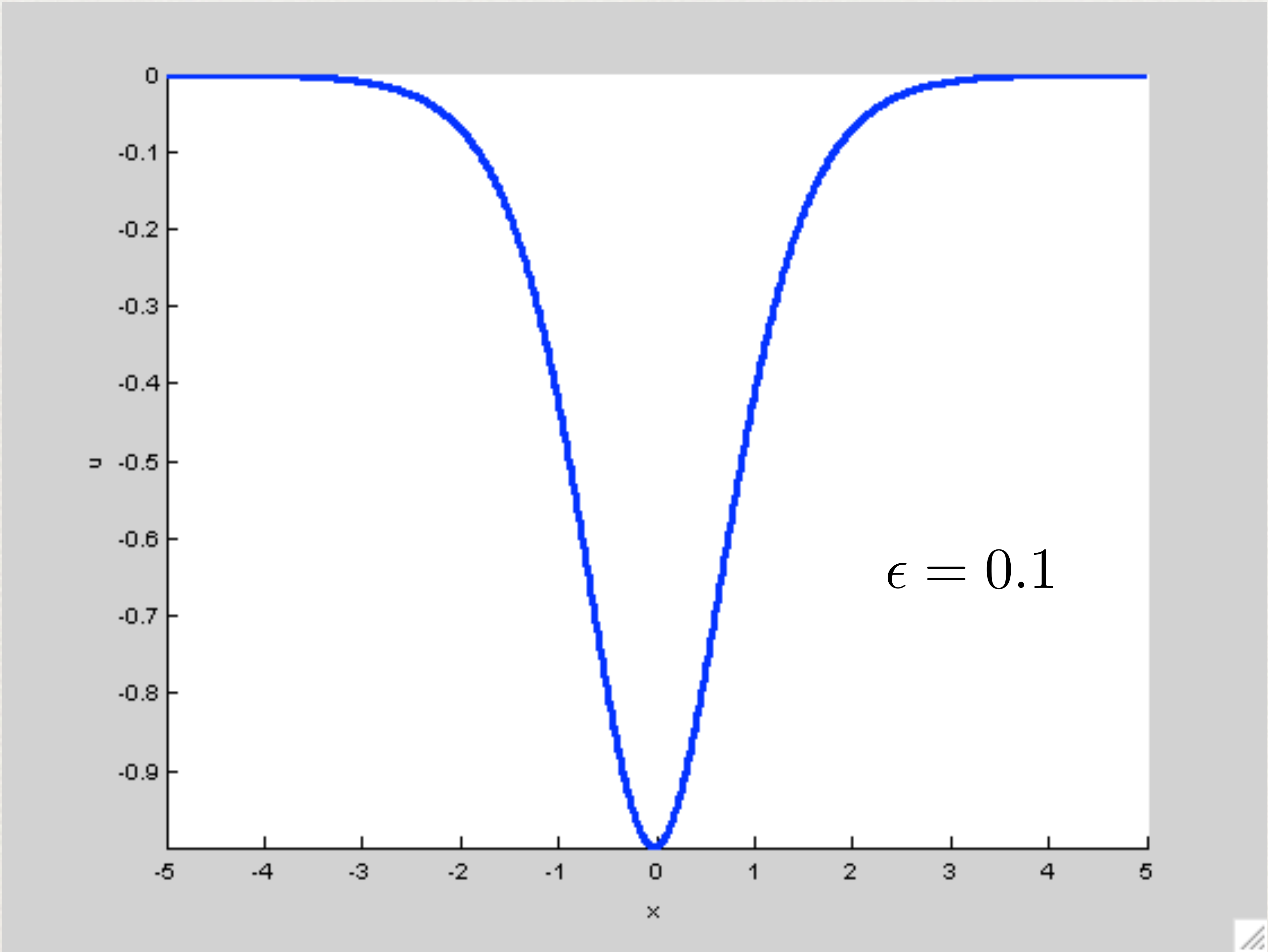
Dissipative regularization

$$u_t + 6uu_x = \epsilon u_{xx}$$

$\epsilon = 0.1$

Dissipative regularization

$$u_t + 6uu_x = \epsilon u_{xx}$$



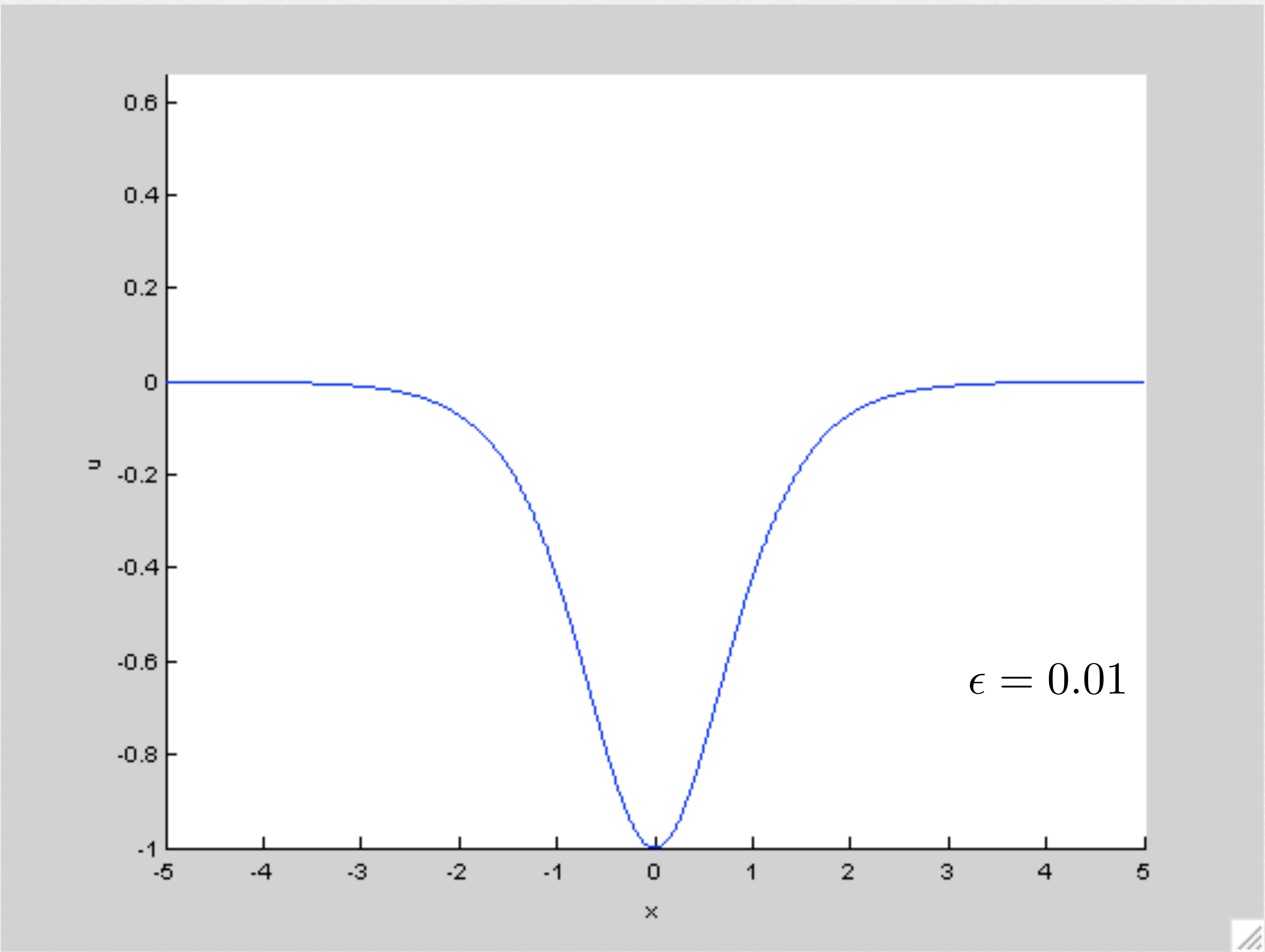
Korteweg-de Vries equation

$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0 \qquad u_0 = -\mathrm{sech}^2 x$$

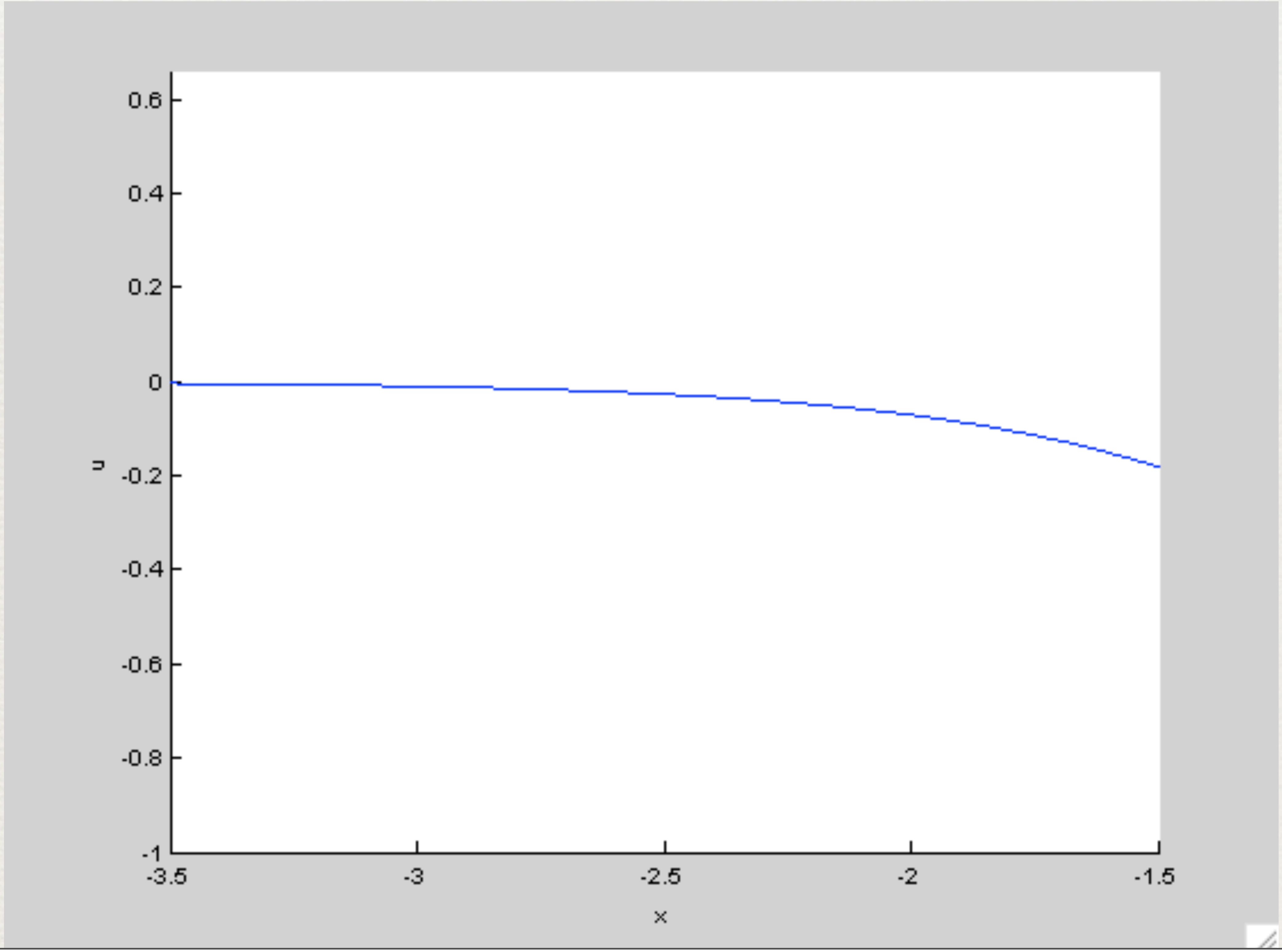
$\epsilon = 0.01$

Korteweg-de Vries equation

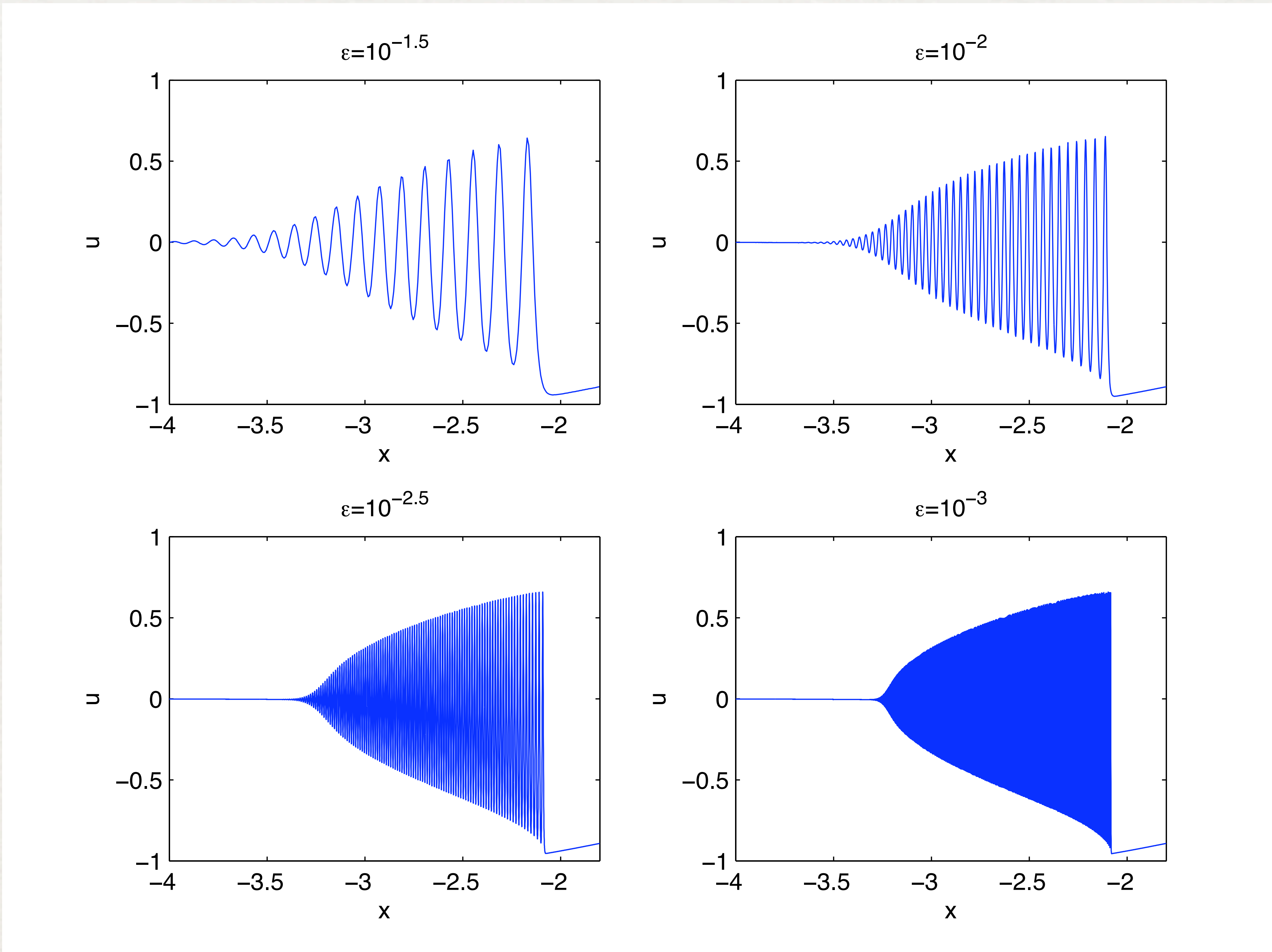
$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0 \qquad u_0 = -\text{sech}^2 x$$



Zoom in: oscillatory zone



Different values of ε



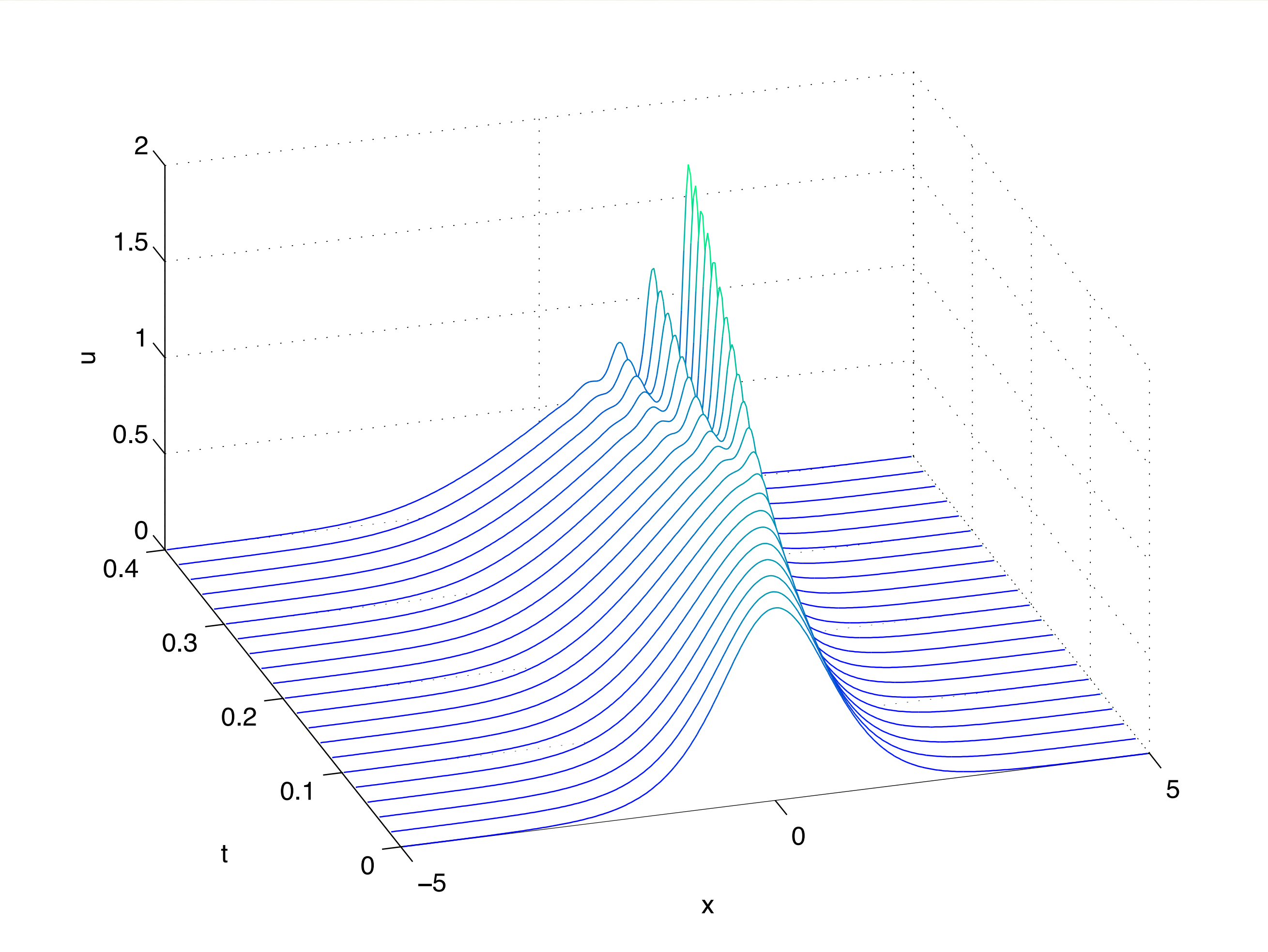
$t = 0.4$

Solitons

- traveling localized wave

$$u = \frac{c}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right), \quad c = const$$

- for large t localized initial data decompose into solitons and radiation



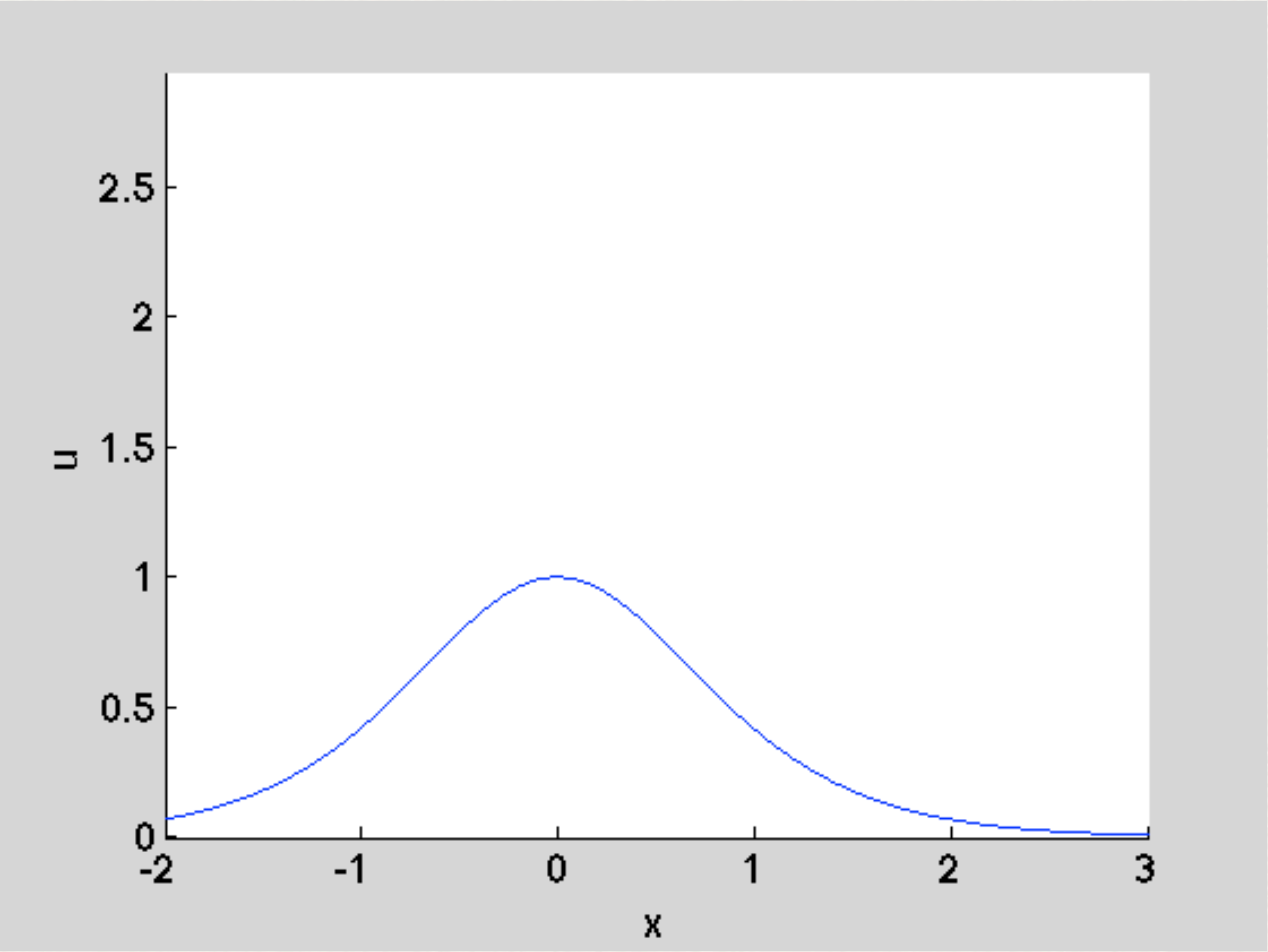
$$u_0 = \text{sech}^2 x, \quad \epsilon = 0.1$$

generalized KdV

$$u_0 = \text{sech}^2 x, \quad \epsilon = 0.01$$

generalized KdV

$$u_0 = \mathrm{sech}^2 x, \quad \epsilon = 0.01$$



Blow-up

- generalized KdV equation

$$u_t + u^p u_x + \epsilon^2 u_{xxx} = 0, \quad p \in \mathbb{N}$$

- $$u_t + \epsilon^2 u_{xxx} = 0$$

  (linear) and

  $$u_t + u^p u_x = 0$$

  (shocks) do not have blow-up of the L_∞ norm of u

- for $p < 4$: global existence in time,
  for $p = 4$: finite time blow-up (Martel, Merle, Raphaël: rescaled soliton),
  for $p > 4$: finite time blow-up, no theory yet.

Spectral Methods

References

[1] Boyd, J. P., *Chebyshev and Fourier Spectral Methods*, Dover, Mineola, NY, 2001. http://www-personal.engin.umich.edu/ jpboyd/.

[2] Canuto, C., Hussaini, M. Y. Quarteroni A. et Zang, T. A., *Spectral Methods in Fluid Dynamics*, Springer-Verlag, Berlin, 1988.

[3] Fornberg, B., *A practical guide to pseudospectral methods*, (Cambridge University Press, Cambridge 1996)

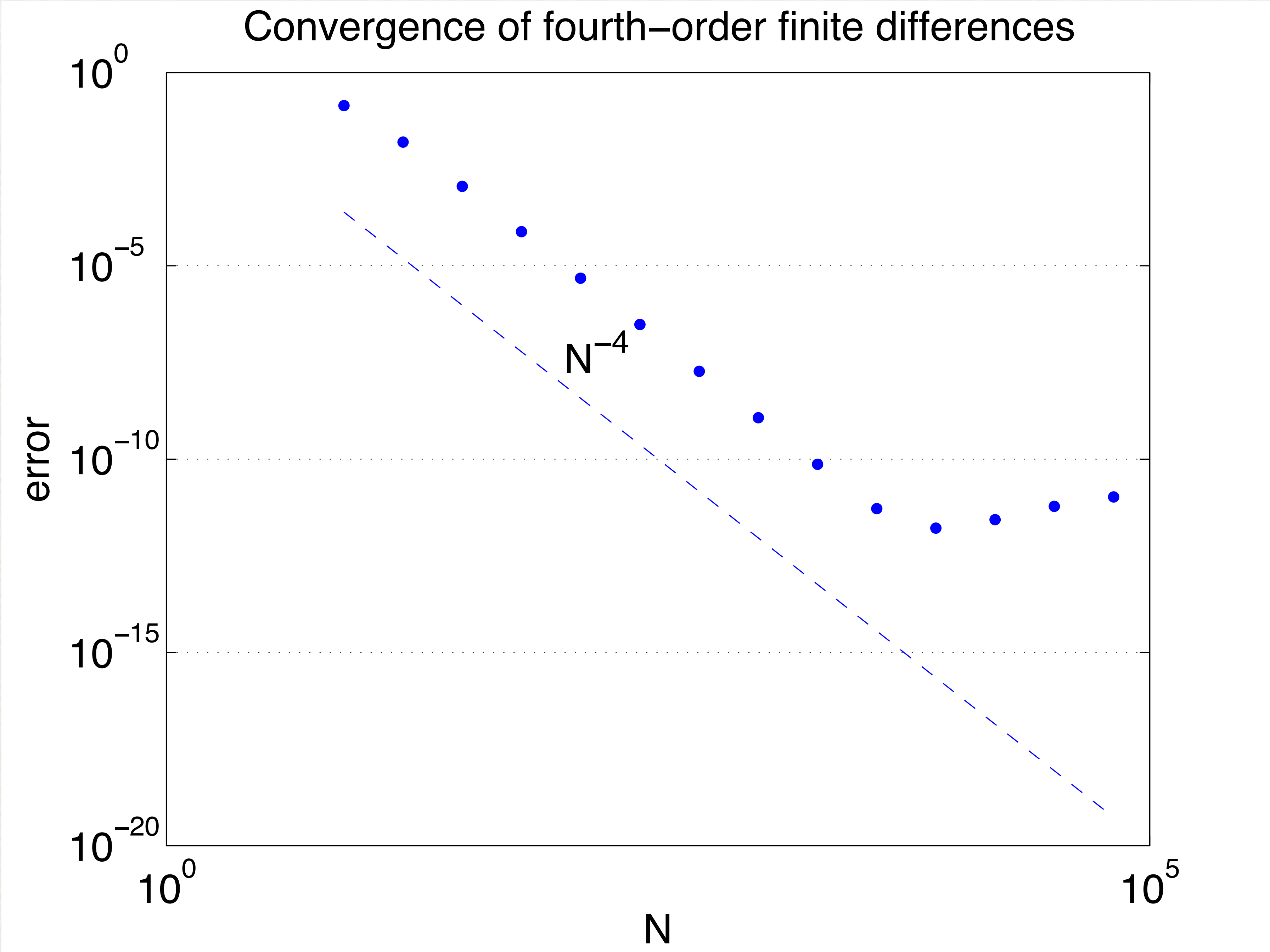
[4] Trefethen, L. N., *Spectral Methods in MATLAB*, SIAM, Philadelphia, PA, 2000.

[5] http://people.maths.ox.ac.uk/trefethen/spectral.html

Introduction

- numerical solution of a PDE: approximate solution with finite precision on a numerical grid (discretization)
- methods:
  - finite differences (1950s): local polynomials of low order
  - finite elements (1960s): local smooth functions
  - spectral methods (1970s): global smooth functions

Example: Fourth order differentiation for $u = \exp(\sin(x))$

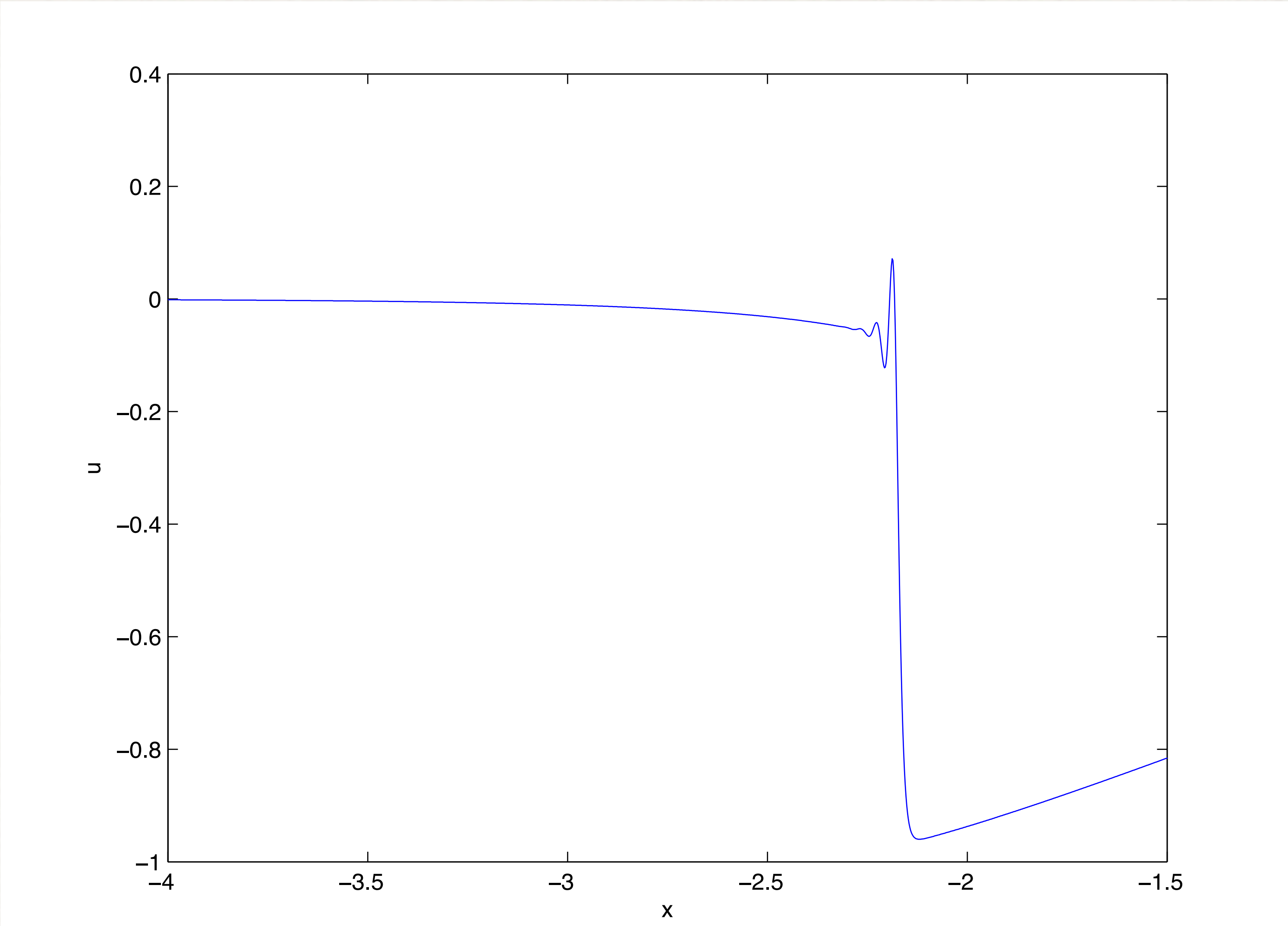


KdV-Burgers equation

$$u_t + 6uu_x + \epsilon^2 u_{xxx} = \delta u_{xx}, \quad u_0 = -\text{sech}^2 x$$

critical value: $\epsilon = 0(\delta)$

C. Kondo and P.G. LeFloch, *Zero diffusion-dispersion limits for hyperbolic conservation laws*, SIAM J. Math. Anal. **33** (2002), 1320–1329.



$\delta = 10^{-2}$

$\epsilon = 10^{-4}$

$t = 0.4$

Spectral differentiation via FT

- periodic function: $v_{j+mN} = v_j$, $j, m \in \mathbb{Z}$; spacing $h = \frac{2\pi}{N}$, choose N even. In this case the wave numbers k, $k \in [-\pi/h, \pi/h]$, are also discrete since e^{ikx} only periodic for integer wave numbers: discrete Fourier transform (DFT)

$$\hat{v}_k = h \sum_{j=1}^{N} e^{-ikx_j} v_j, \quad k = -\frac{N}{2} + 1, \ldots, \frac{N}{2},$$

$$v_j = \frac{1}{2\pi} \sum_{k=-N/2+1}^{N/2} e^{\,ikx_j} \hat{v}_k, \quad j = 1, \ldots, N$$

for spectral differentiation symmetrization useful:

$$v_j = \frac{1}{2\pi} \sum_{k=-N/2+1}^{N/2-1} e^{\,ikx_j} \hat{v}_k + \frac{1}{4\pi} \left(\hat{v}_{-N/2} e^{-ix_j N/2} + \hat{v}_{N/2} e^{ix_j N/2} \right), \quad j = 1, \ldots, N$$

Periodicity and DFT

- equivalent approach via DFT:
  - given v, compute $\hat{v}$
  - define $\hat{w}_k = ik\hat{v}_k$ ($\hat{w}_{N/2} = 0$)
  - compute w
- fast Fourier transform (FFT), 1965 by Cooley and Tuckey, 1805 by Gauss: if N is a product of primes, $O(N \log N)$ floating point operations. Matlab: wave vector $0, 1, \ldots, N/2, -N/2+1, -N/2+2, \ldots, -1$; works on complex vector (factor 2 lost for real vectors).

p5

Smoothness and spectral accuracy

- convergence of Fourier series (Dirichlet): if u is piecewise differentiable, then $\sum_{k\in\mathbb{Z}} \hat{u}(k)e^{ikx}$ converges to u. Consequently if $u \in C^m$, the Fourier series converges for $u^{(m-1)}$, i.e., $\sum_{k\in\mathbb{Z}} k^{m-1}\hat{u}(k)e^{ikx}$. The smoother a function in physical space, the more it is localized in Fourier space.

- **theorem 1** *Smoothness of a function and decay of its Fourier transform* *Let* $u \in \mathrm{L}^2(\mathbb{R})$ *have Fourier transform* $\hat{u}(k)$.

  1. *If* $u \in C^{p-1}(\mathbb{R})$ *for some* $p \geq 0$ *and has a pth derivative of bounded variation, then*

  $$\hat{u}(k) = O(|k|^{-p-1}) \text{ as } |k| \to \infty \; .$$

  2. *If* $u \in C^{\infty}(\mathbb{R})$, *then*

  $$\hat{u}(k) = O(|k|^{-m}) \text{ as } |k| \to \infty$$

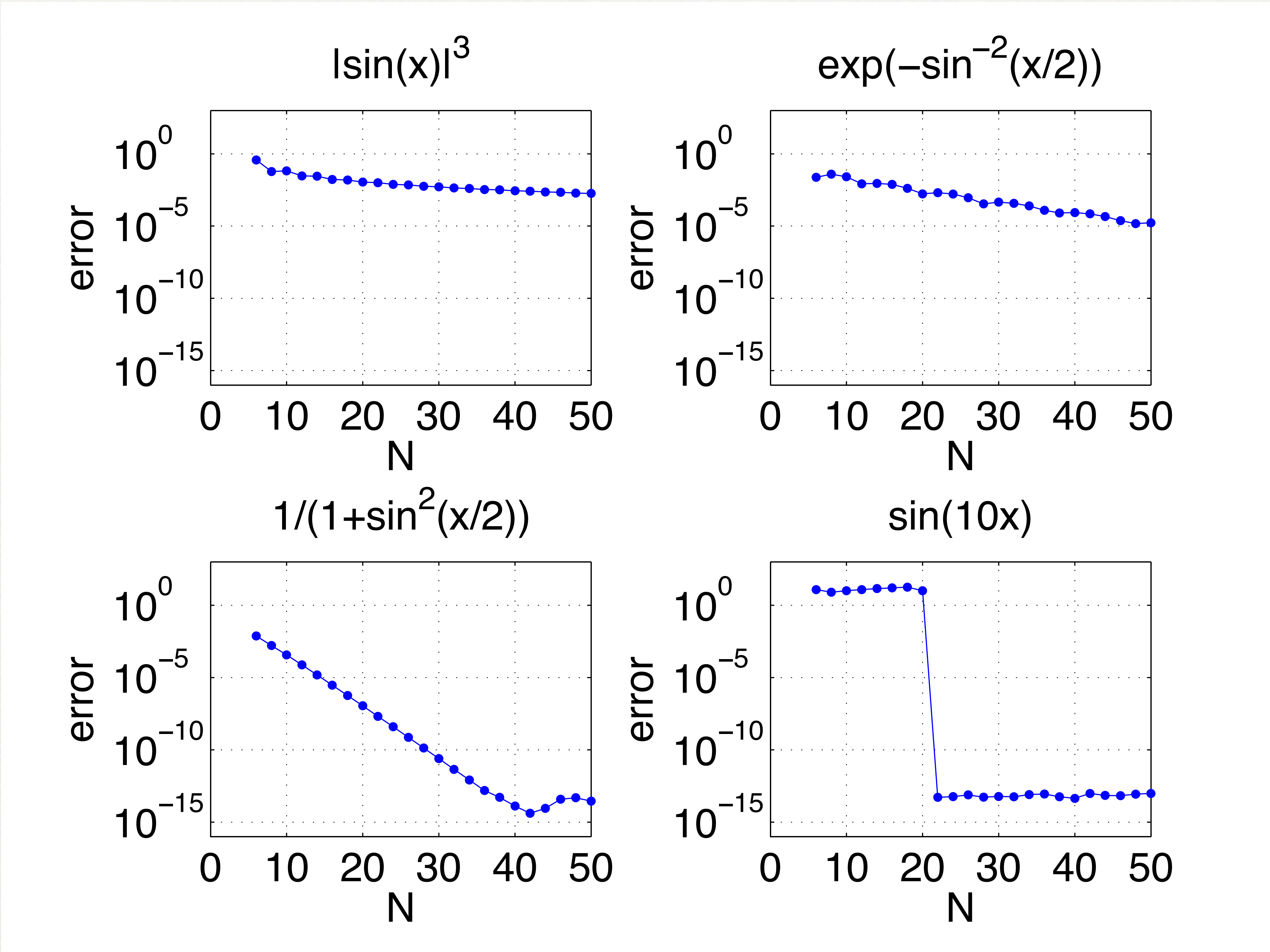
  for every $m \geq 0$. *The converse also holds.*

  3. *If there exist* $a, c > 0$ *such that* u *can be extended to an analytic function in the complex strip* $|\Im z| < a$ *with* $||u(\cdot + iy)||$ *uniformly for all* $y \in [-a, a]$, *where* $||u(\cdot + iy)||$ *is the* L^2 *norm along the horizontal line* $\Im z = y$, *then* $u_a \in L^2(\mathbb{R})$, *where* $u_a(k) = e^{a|k|}\hat{u}(k)$. *The converse also holds.*

  4. *If* u *can be extended to an entire function (i.e., analytic throughout the complex plane) and there exists* $a > 0$ *such that* $|u(z)| = o(e^{a|z|})$ *as* $|z| \to \infty \; \forall z \in \mathbb{C}$, *then* $\hat{u}$ *has compact support in* $[-a, a]$.

p7

Example: differentiation with fft

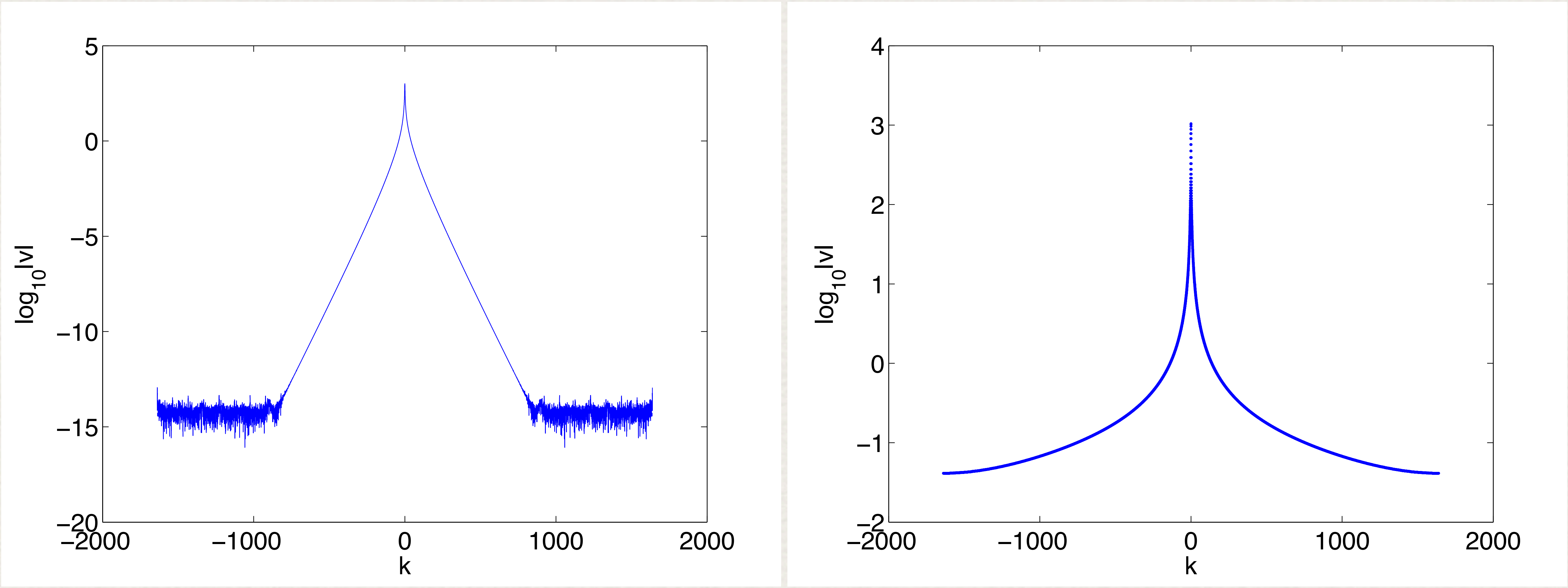


Fourier coefficients, Hopf

- singularities in the complex plane: $u(z) \sim (z - z_j)^{\mu_j}$
- $\hat{u} \sim k^{-(\mu_j+1)} e^{-k\delta} e^{ik\Re z_j}$ for $k \to \infty$,

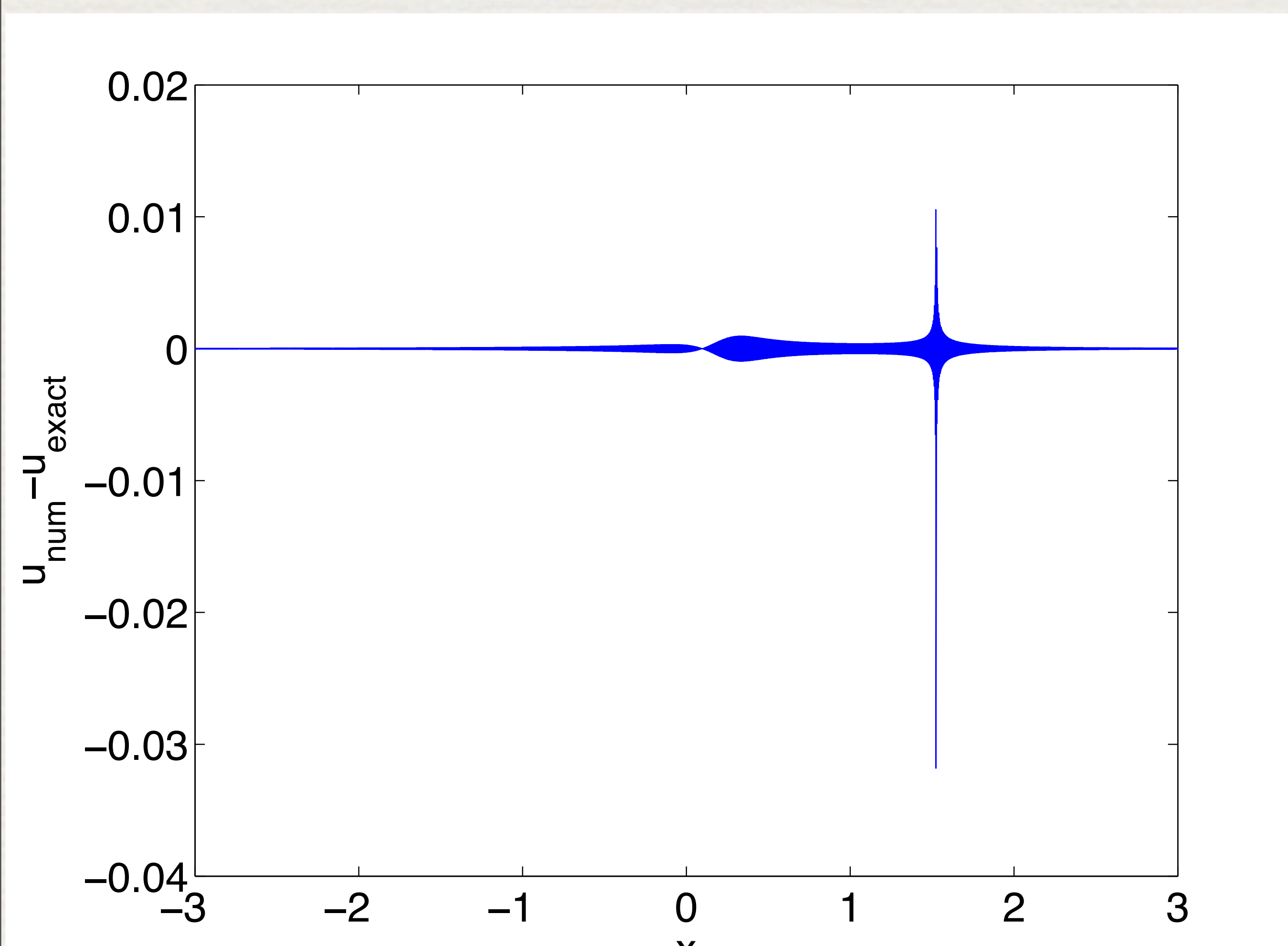
oscillations if several singularities are present

$t \ll t_c$ $\quad u_0 = -\mathrm{sech}^2 x \quad$ $t = t_c$

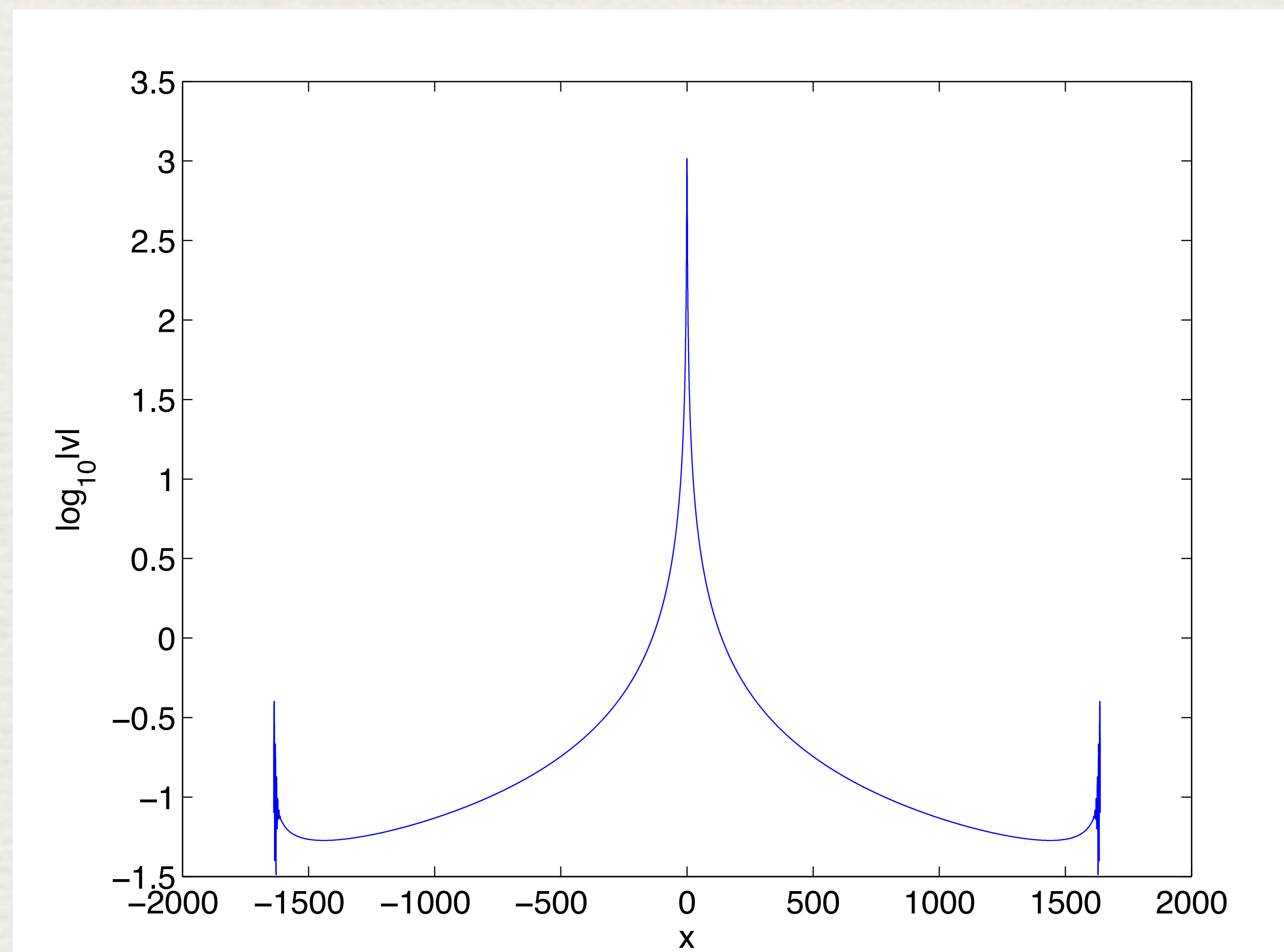


Numerical solution

Difference between numerical and exact solution

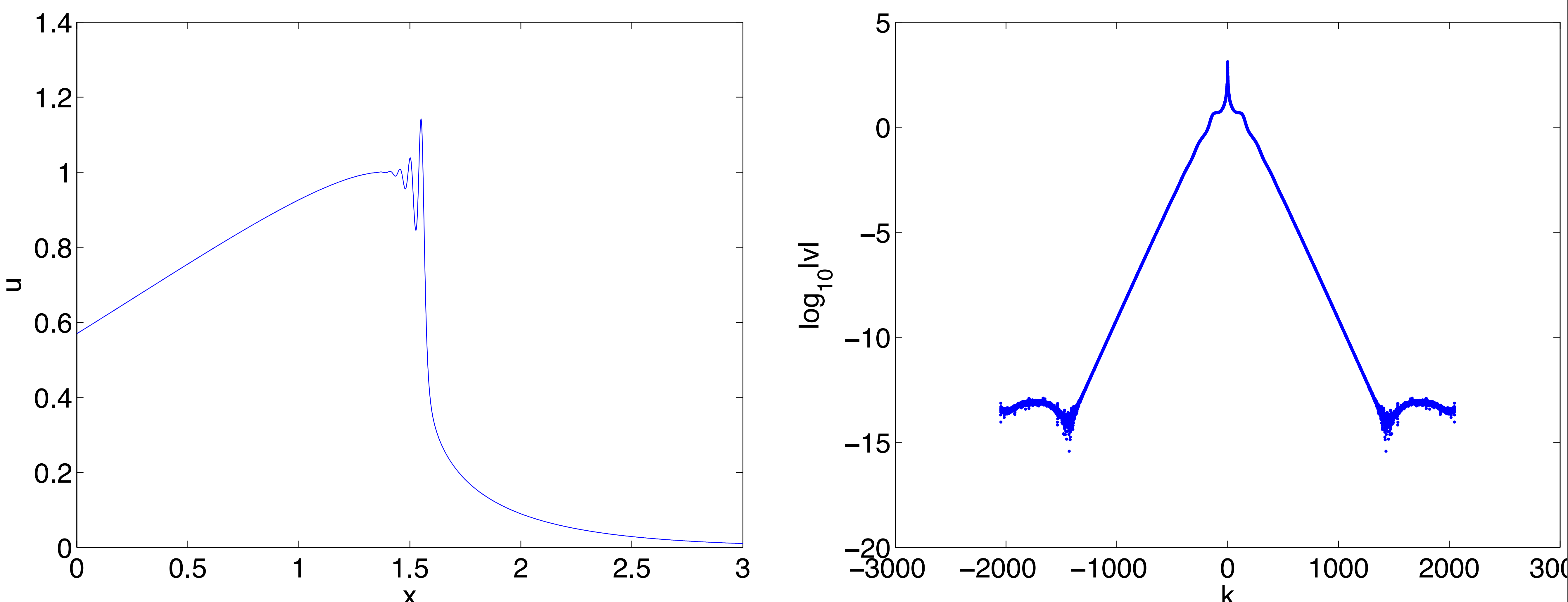


Fourier coefficients of the numerical solution



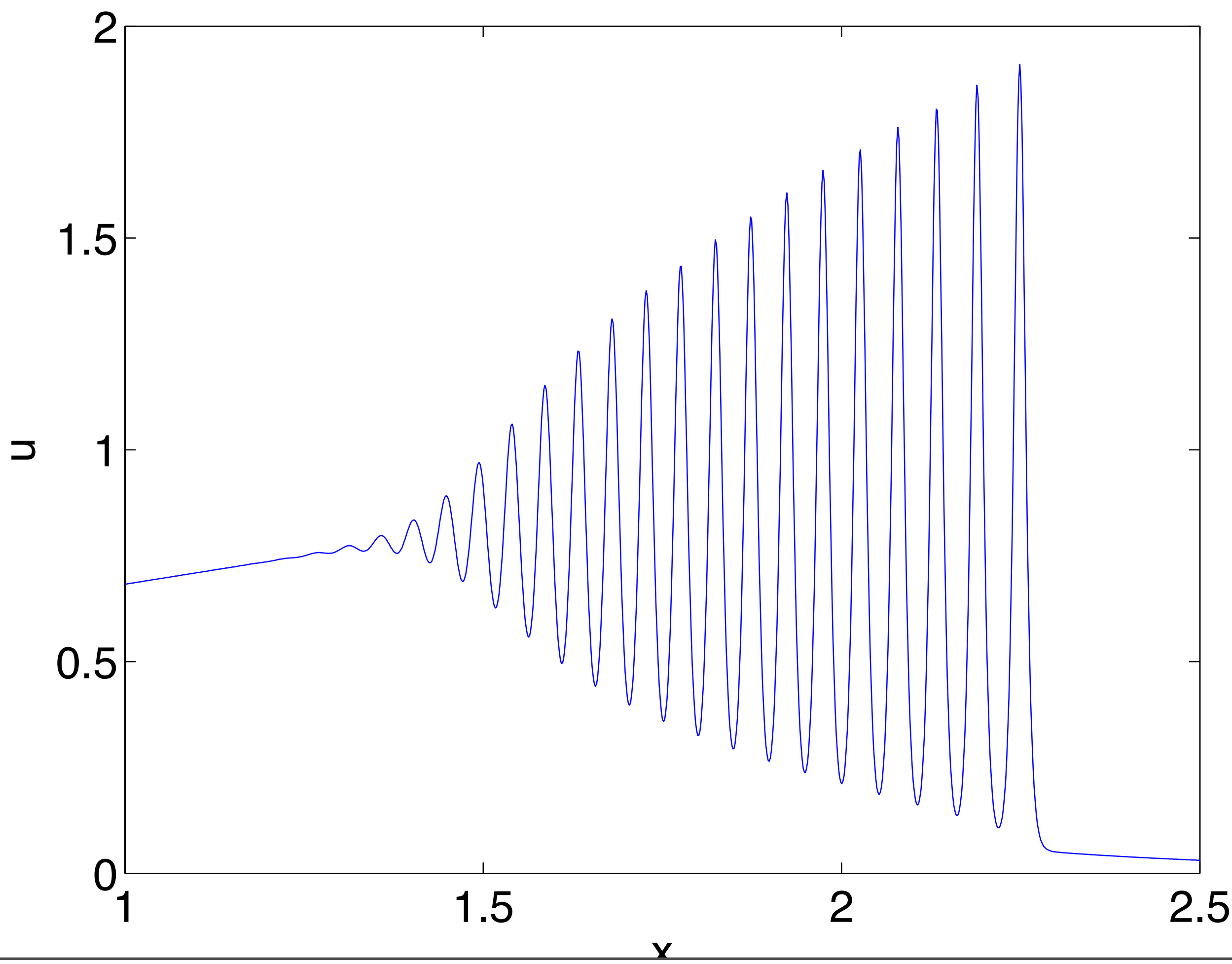
KdV equation

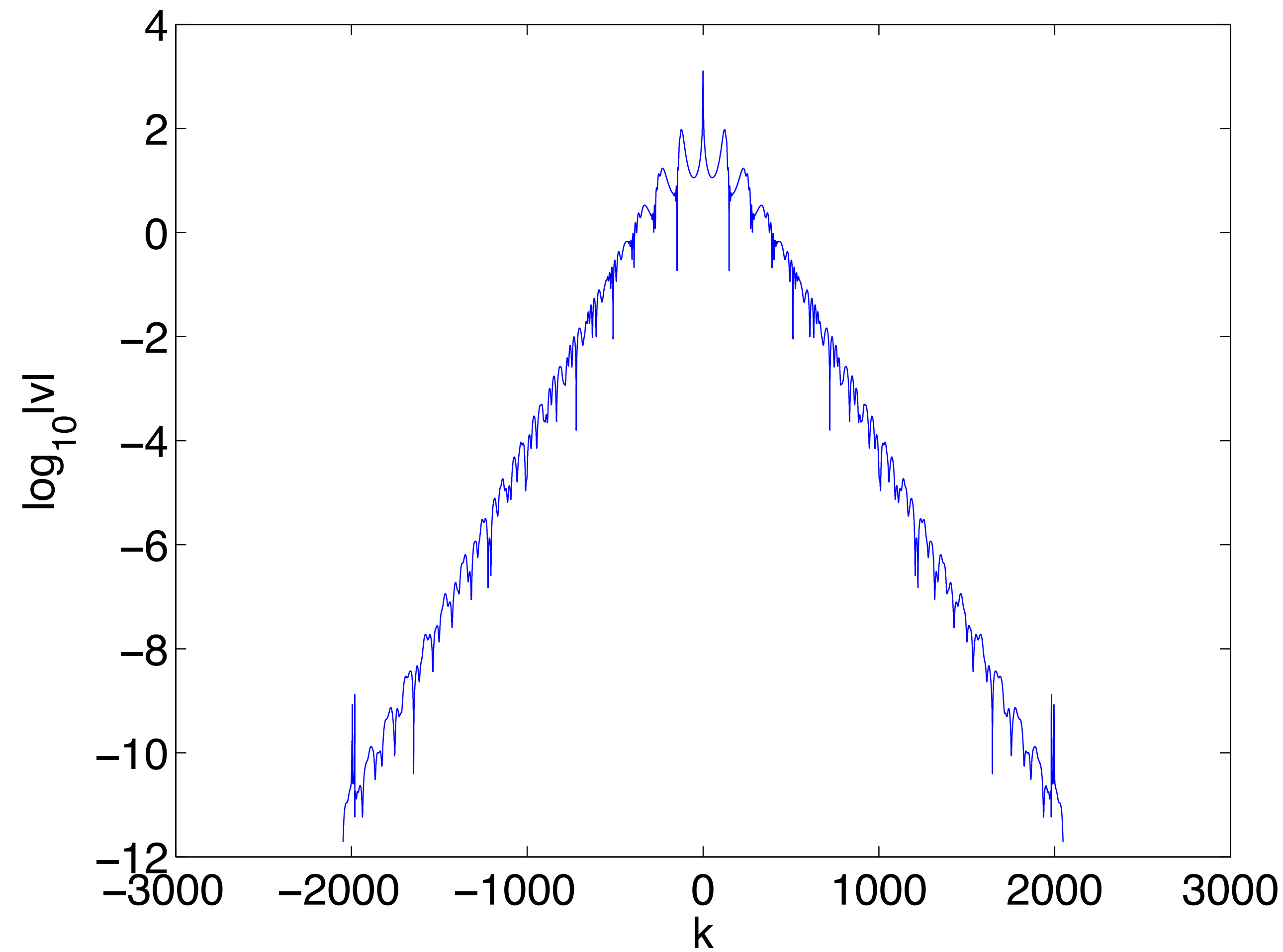
$u_0 = \operatorname{sech}^2 x$ $\quad$ $t \sim t_c \quad \epsilon = 0.01$



KdV equation

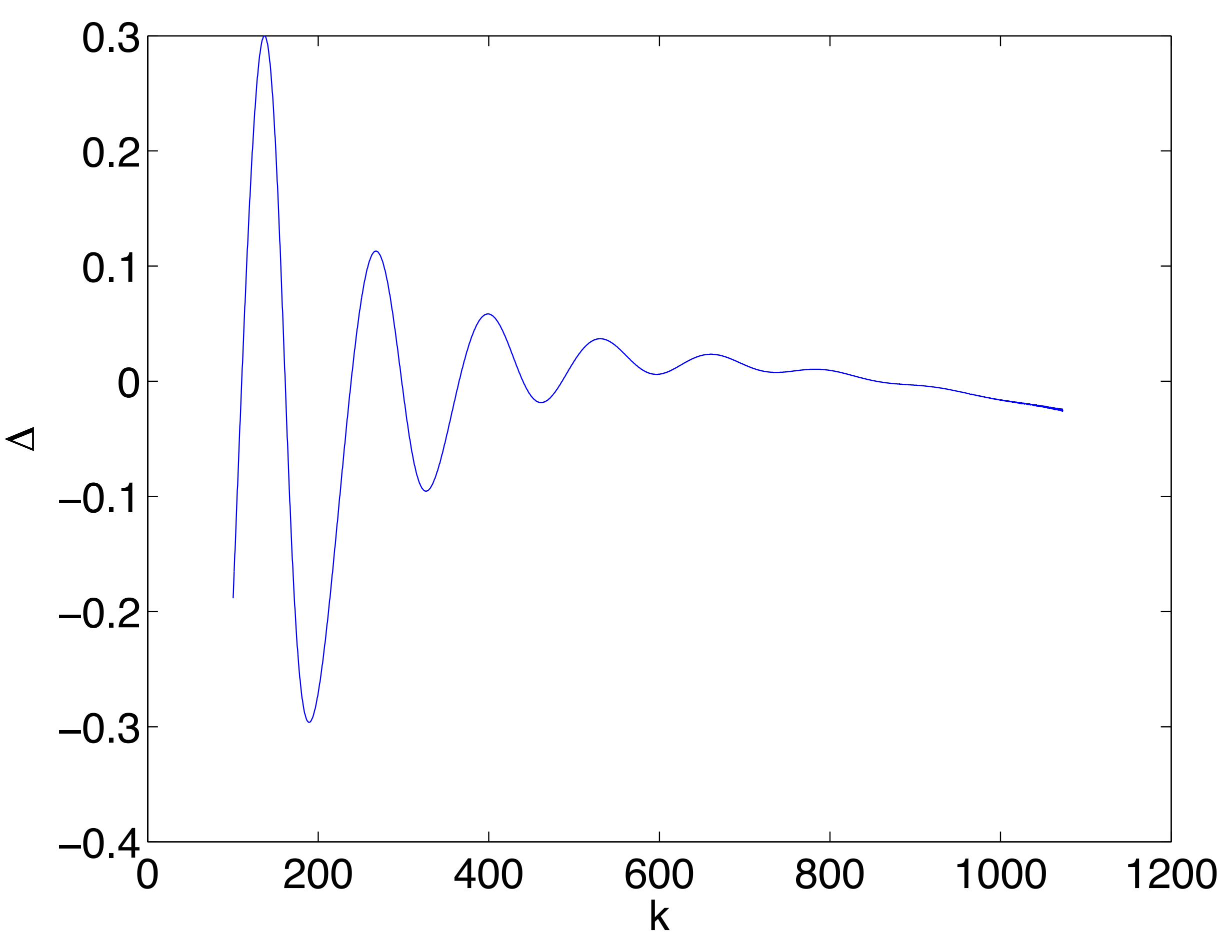
$u_0 = \operatorname{sech}^2 x$ $\quad t \gg t_c$ $\quad \epsilon = 0.01$



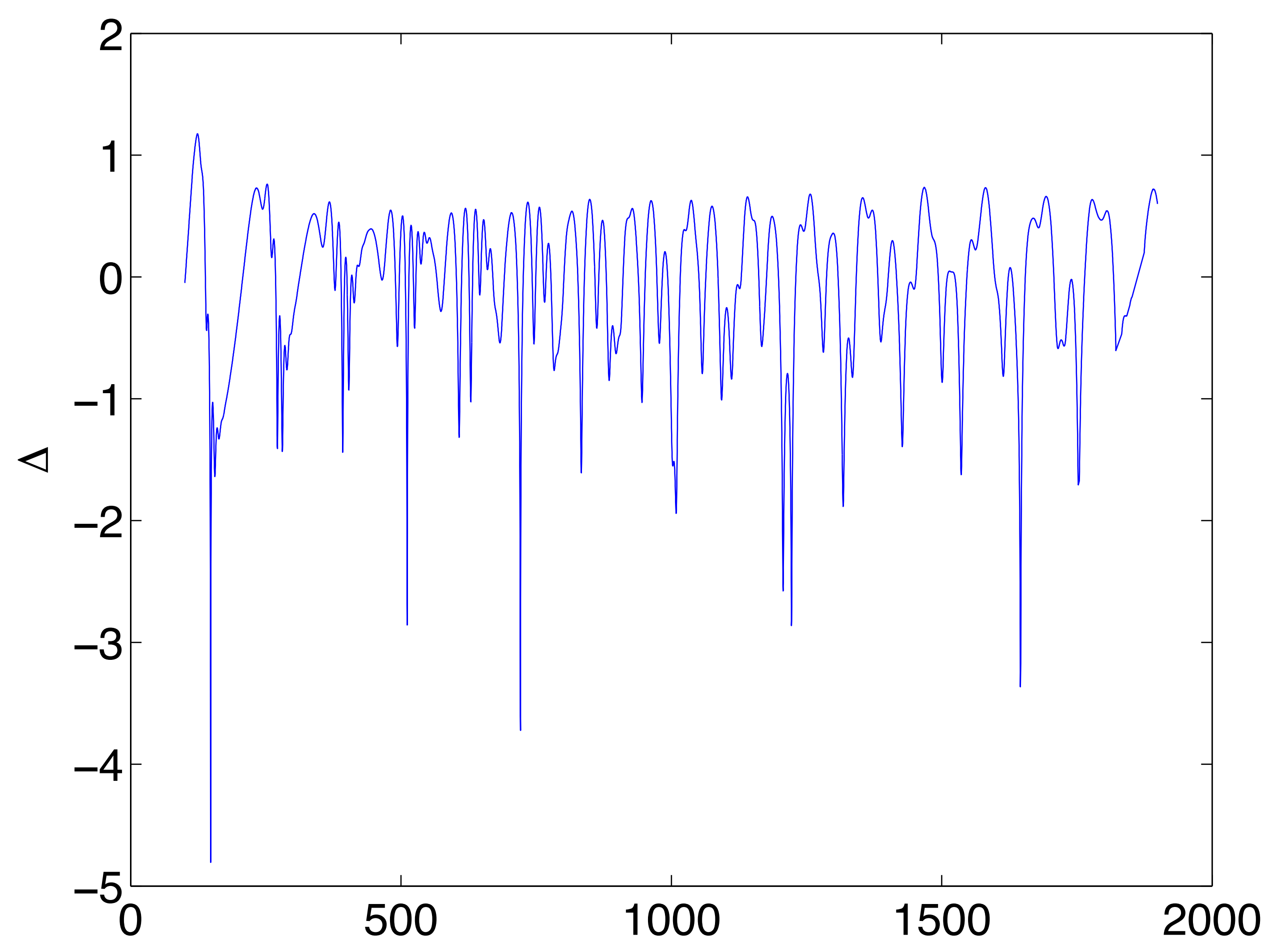


Fitted Fourier coefficients

$t \sim t_c$

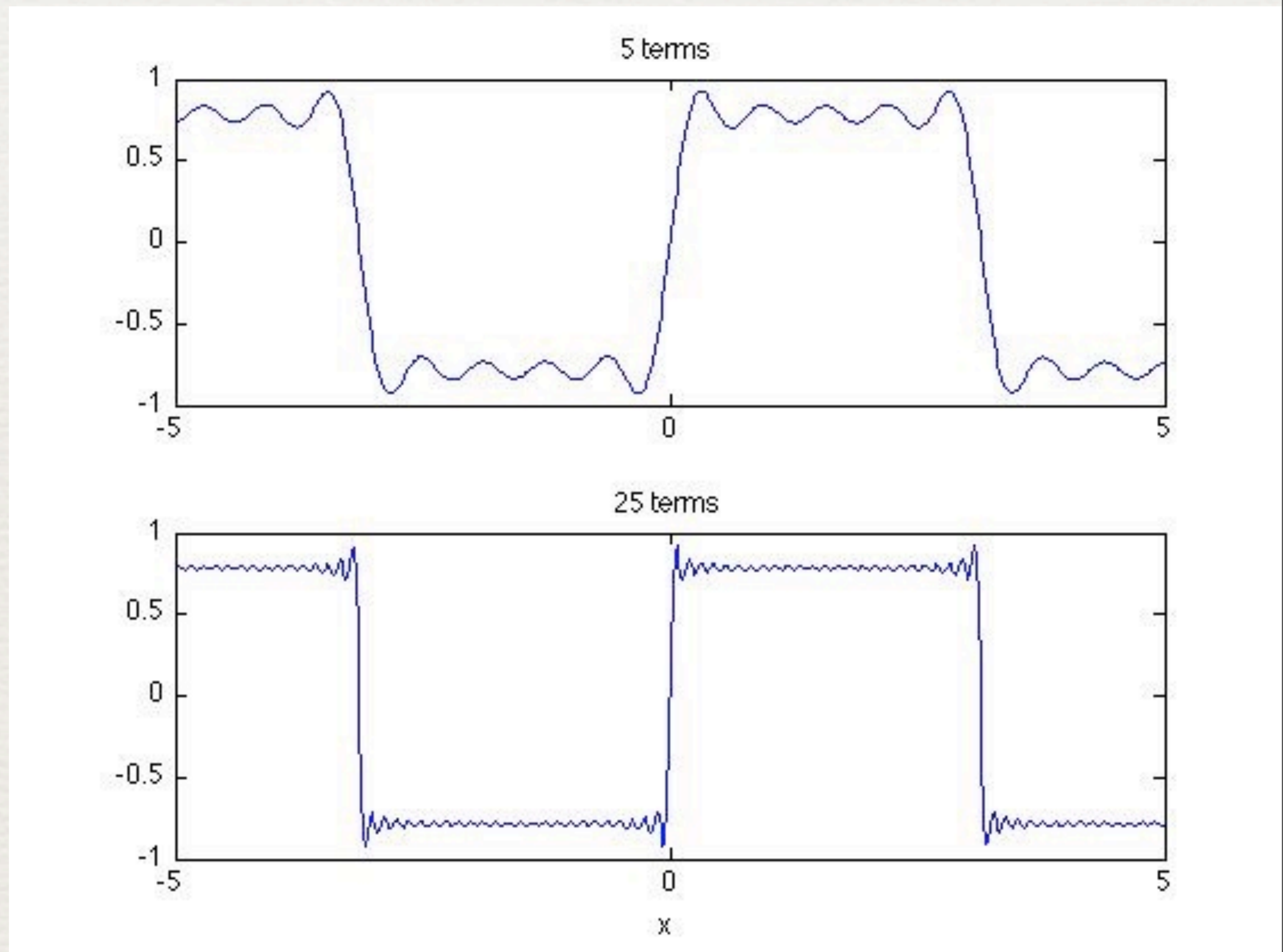


$t \gg t_c$



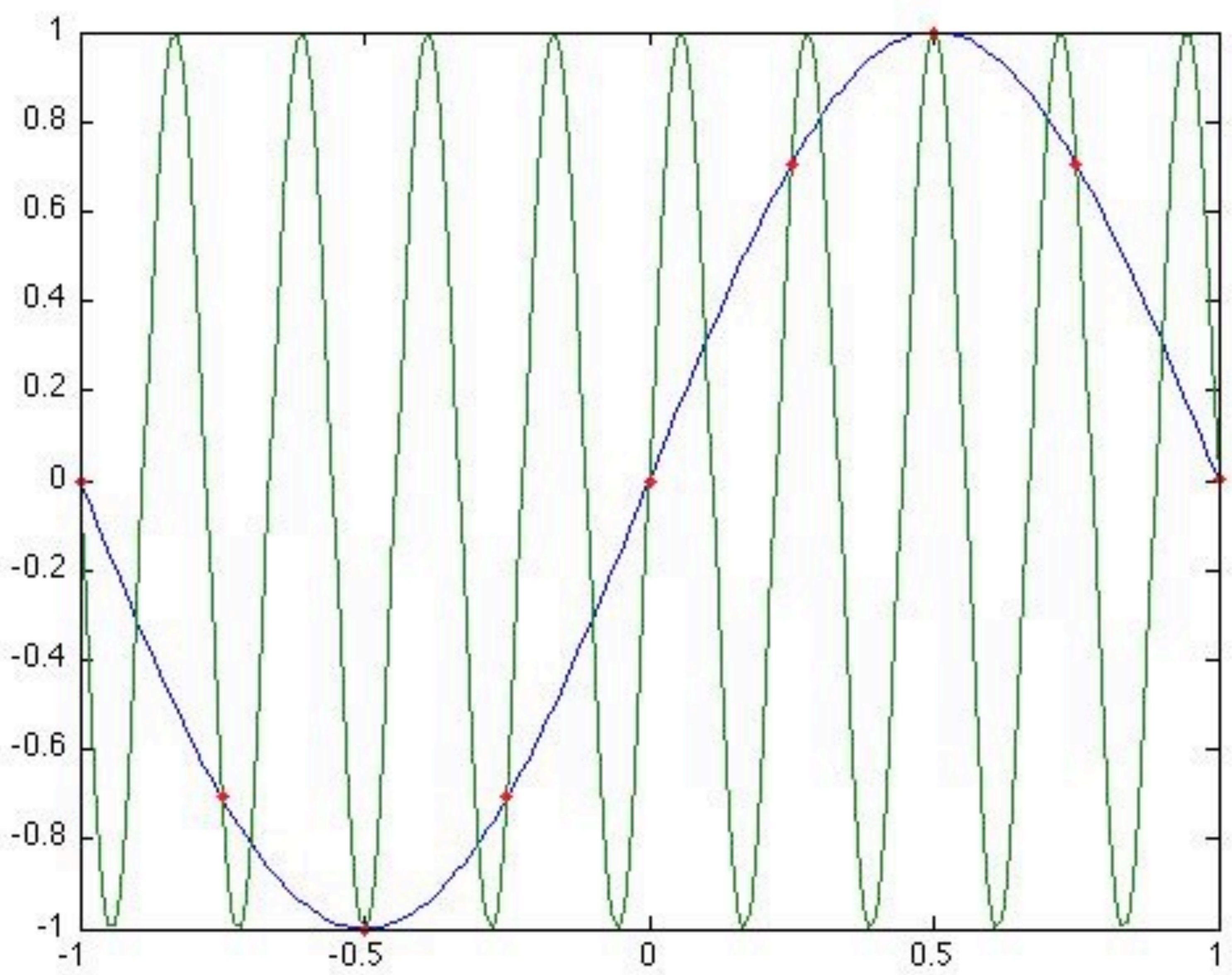
Gibbs phenomenon

- convergence of Fourier series at discontinuities of the function: strong oscillations, `overshooting' by roughly 9%
- the Fourier series for the step function (5 respectively 25 terms)



Aliasing

- $e^{ik_1 x} = e^{ik_2 x}$ implies $k_1 = k_2$ for $x \in \mathbb{R}$, but only $k_1 - k_2 = \frac{2\pi}{h} n$, $n \in \mathbb{Z}$, for $x \in h\mathbb{Z}$; the additional values for $n \neq 0$ are called 'aliases' of k.



$\sin(\pi x)$ and $\sin(9\pi x)$ equivalent on the grid $\frac{1}{4}\mathbb{Z}$

- **theorem 2** *Aliasing formula*
Let $u \in L^2(\mathbb{R})$ have a first derivative with bounded variation, and let v be the grid function on $h\mathbb{Z}$ defined by $v_j = u(x_j)$. Then $\forall k \in [-\pi/h, \pi/h]$,

$$\hat{v}(k) = \sum_{j \in \mathbb{Z}} \hat{u}(k + 2\pi j/h)$$

This implies that

$$\hat{v}(k) - \hat{u}(k) = \sum_{j \in \mathbb{Z}, j \neq 0} \hat{u}(k + 2\pi j/h).$$

Smoothness can be related to the error $\hat{v}(k) - \hat{u}(k)$ (*aliasing error*).

p8

Fractional KdV equations

- in contrast to gKdV, lower the dispersion: fractionary KdV (fKdV) equation

$$u_t + uu_x - D^\alpha u_x = 0, \quad \widehat{D^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi)$$

  $\alpha = 2$: KdV, $\alpha = 1$: Benjamin-Ono

- Solitons $u(x,t) = Q_c(x - ct)$ (Linares, Pilod, Saut 2013)

$$D^\alpha Q_c + cQ_c - \frac{1}{2}Q_c^2 = 0,$$

  solitons exist for $\alpha \geq 1/3$, not for $\alpha < 1/3$, algebraic decay for $x \to \infty$

- no blow-up expected for $\alpha > 1/2$

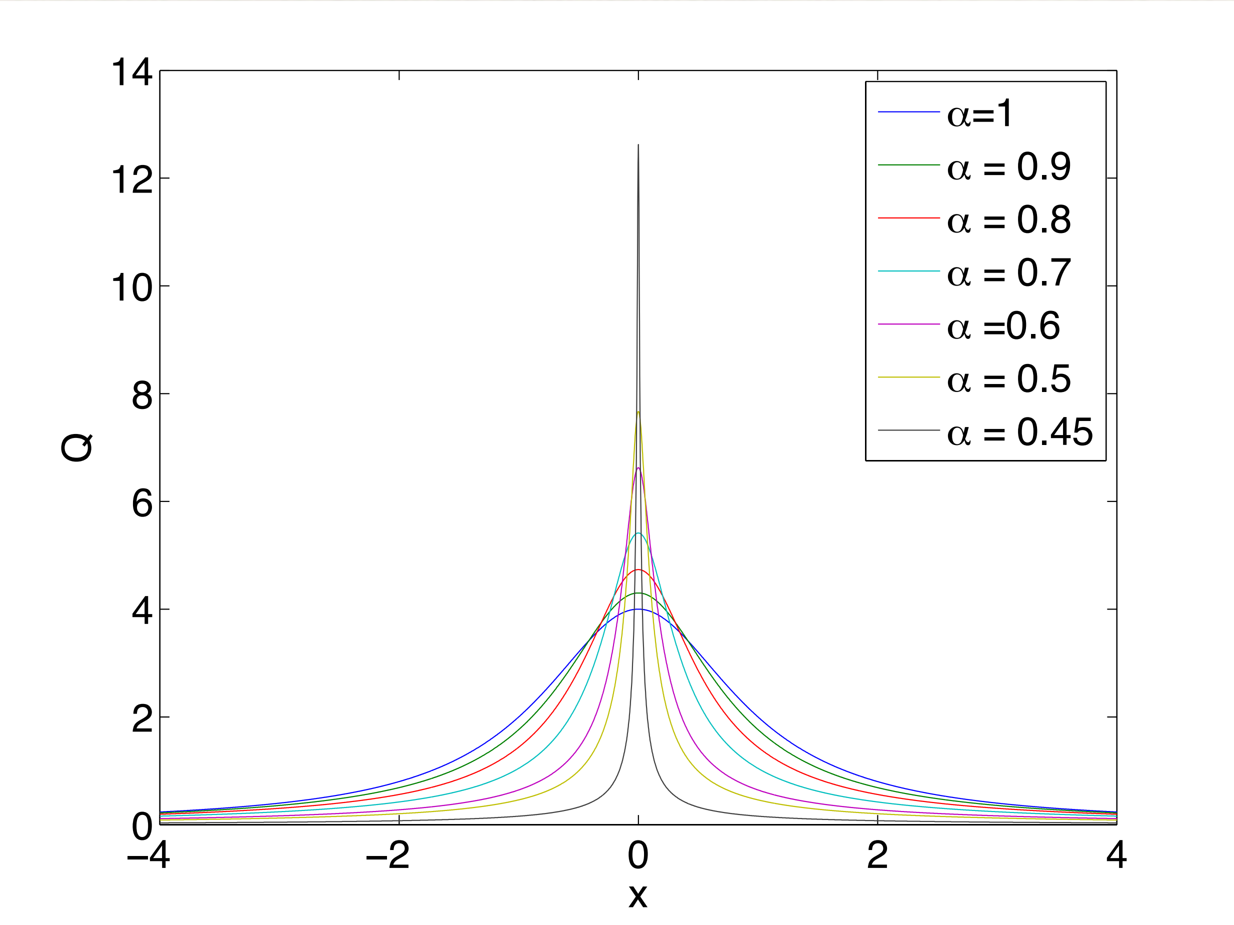
C. Klein and J.-C. Saut, *A numerical approach to blow-up issues for dispersive perturbations of Burgers' equation*, Physica D: Nonlinear Phenomena (2015), 46-65, 10.1016/j.physd.2014.12.004

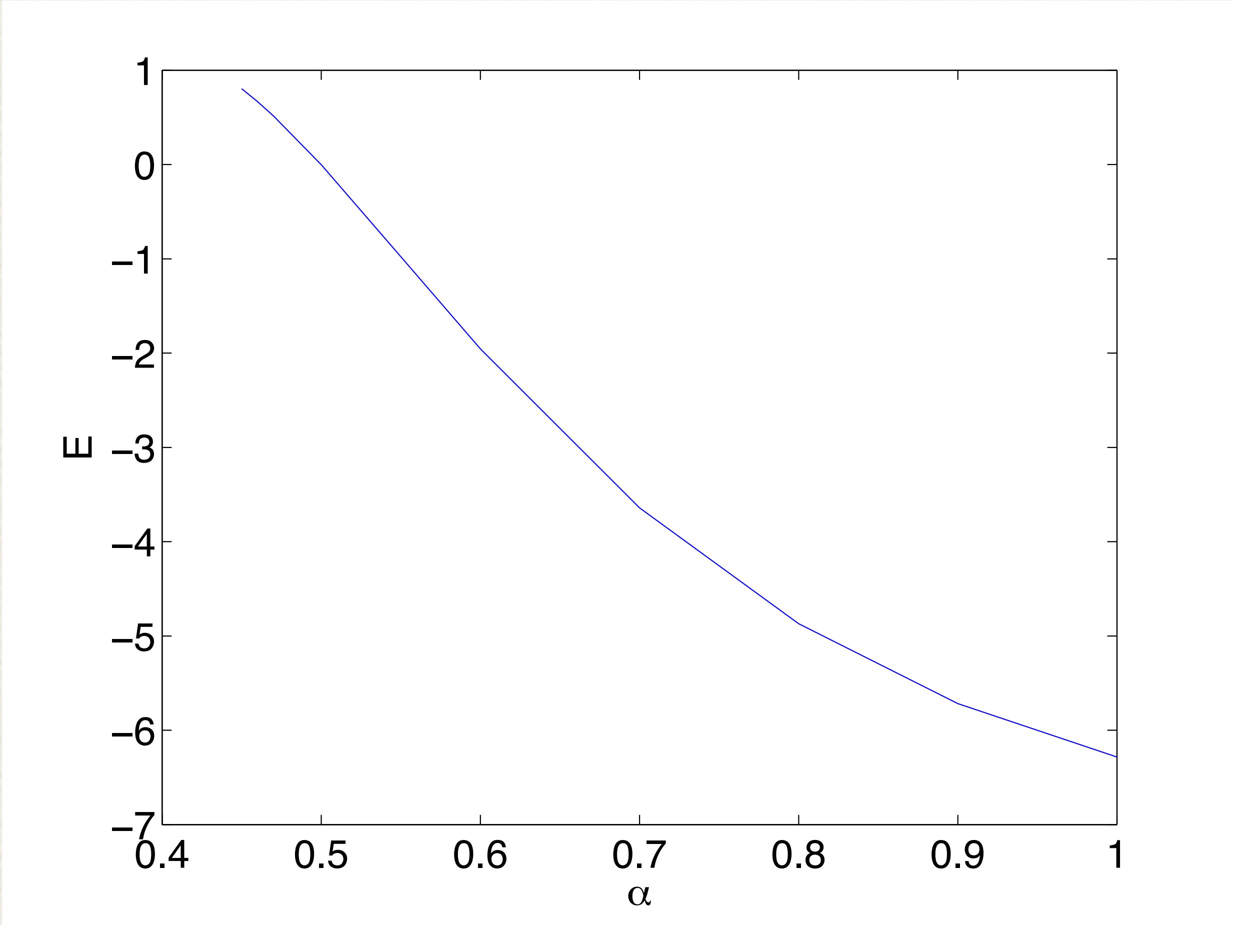
Solitary waves

- travelling wave solutions $u = Q(x - ct)$, rescalings of x and u to put $c = 1$,

$$D^\alpha Q_c + cQ_c - \frac{1}{2}Q_c^2 = 0$$

- Newton-Krylov method to construct the solutions numerically for decreasing α starting from the known solution to BO





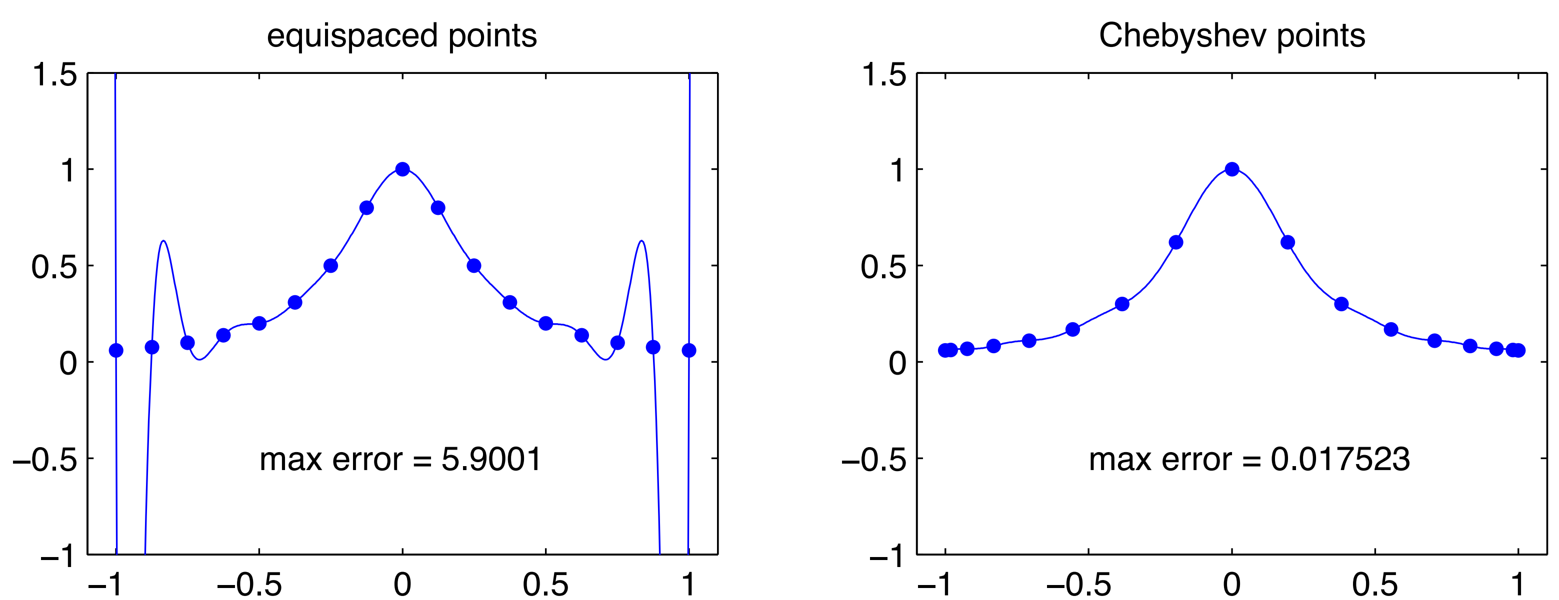
Finite intervals

- Periodic continuation of non-periodic functions on a finite interval: discontinuous functions on the real line, Gibbs phenomenon.
- Advantage of spectral methods lost. Therefore polynomial interpolation, not trigonometric.
- Uniform grids: Runge phenomenon

p9

Runge phenomenon

interpolation of $1/(1 + 16x^2)$



Spectral differentiation

- Chebyshev grid: projection of equispaced points on the circle ($x \in [-1, 1]$); density $\sim \frac{N}{\pi\sqrt{1-x^2}}$, $O(1/N^2)$ points for $x \sim \pm 1$, $O(1/N)$ for $x \sim 0$.
- Chebyshev (Gauss–Lobatto) points: $x_j = \cos\frac{j\pi}{N}$, $j = 0, 1, \ldots, N$
- polynomial interpolant: unique polynomial of degree $\leq N$ with $p(x_j) = v_j$, $j = 0, 1, \ldots, N$
- approximation of the derivative: $w_j = p'(x_j)$; linear relation, $\vec{w} = D_N \vec{v}$ (N even or odd).

Chebyshev series and FFT

- Chebyshev polynomials $T_n(x) = \cos(n \arccos x)$,
  polynomial of degree N
  unit circle: $|z| = 1$, $x = \Re z = \cos\theta$,
  $x \in [-1, 1]$, $\theta \in \mathbb{R}$, z on unit circle
  Chebyshev series, Fourier series, Laurent series
  $T_n(x) = \Re z^n = \cos n\theta = \frac{1}{2}(z^n + z^{-n})$,
  recursive relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \ldots$$

- Chebyshev polynomials form a basis in the space of polynomials, any polynomial of degree N can be written as a sum of Chebyshev polynomials: $p(x) = \sum_{n=0}^{N} a_n T_n(x)$, $x \in [-1, 1]$.
  Equivalent: Laurent polynomial
  $\mathrm{p}(z) = \frac{1}{2} \sum_{n=0}^{N} a_n (z^n + z^{-n})$, $|z| = 1$
  trigonometric polynomial
  $P(\theta) = \sum_{n=0}^{N} a_n \cos\theta$, $\theta \in \mathbb{R}$

- spectral collocation method:
  $\theta_j = \frac{j\pi}{N}$, $z_j = e^{i\theta_j}$, $x_j = \cos\theta_j$, $j = 0, 1, \ldots, N$

Boundary value problems

- approaches to solve boundary value problems with spectral methods:
  - interpolants that satisfy the boundary conditions (Galerkin method)
  - add supplementary equations corresponding to the boundary conditions: τ-method (more flexible for general boundary conditions); these equations replace the first and the last relations

Multidomain method

- spectral method of 'infinite order', but cond$(D^2) = 0(N^4)$; multi-domains to keep N small
- interval $[x_l, x_r]$ mapped to $[-1, 1]$:

$$x = x_l \frac{1+l}{2} + x_r \frac{1-l}{2}, \quad l \in [-1, 1]$$

- compactified exterior domains (CED): $s = 1/x$ local coordinate,

$$u_{xx} = s^4 u_{ss} + 2s^3 u_s$$

singular for $s = 0$ (compactification with spectral methods first used by Grosch, Orszag 1977, popular in astrophysics)

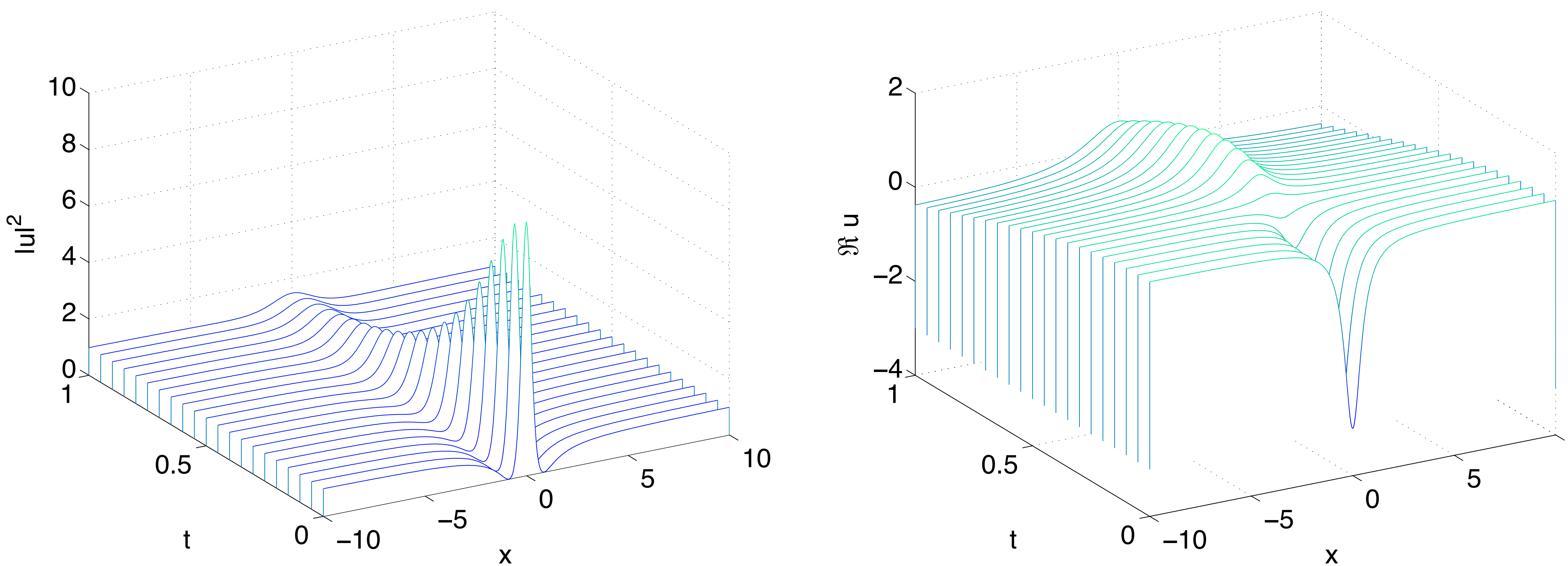
M. Birem and C. Klein, *Multidomain spectral method for Schrödinger equations*, Adv. Comp. Math. DOI 10.1007/s10444-015-9429-9 (2015)

Peregrine breather

- exact solution

$$u_{Per} = \left(1 - \frac{4(1+4it)}{1+4x^2+16t^2}\right) e^{2it}$$

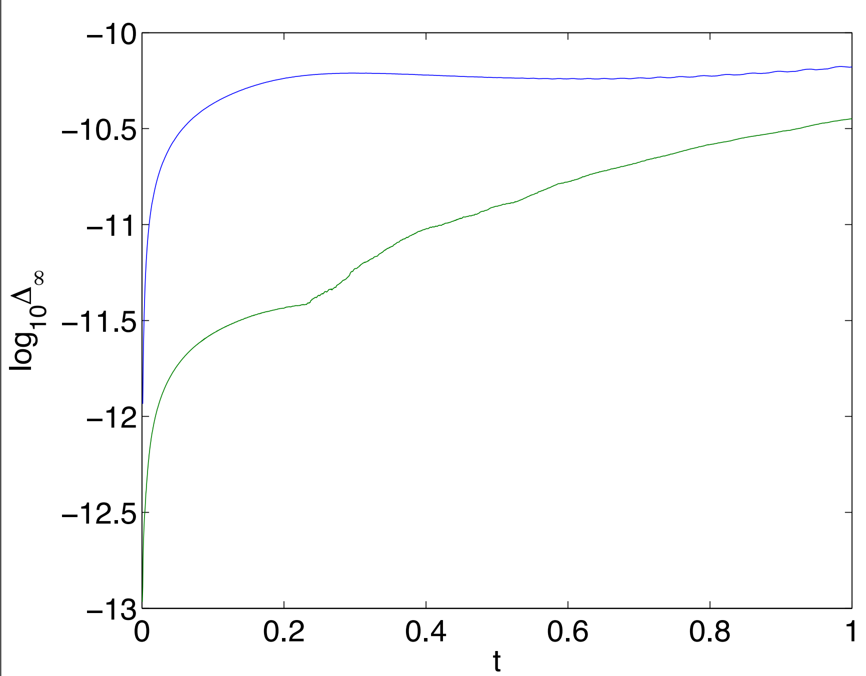
- $|u|$ asymptotically decaying to 1 (both in x and t), maximum three times the asymptotic value

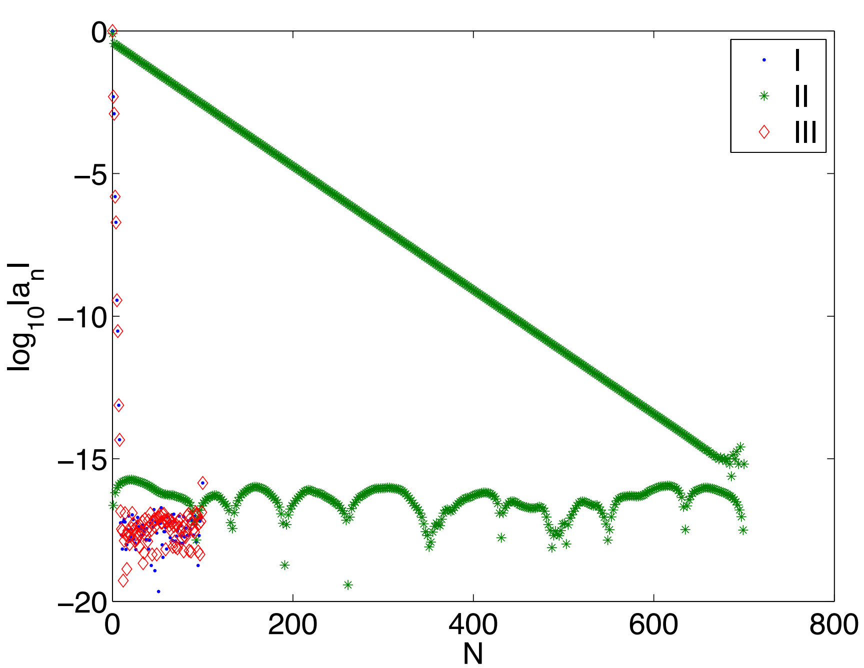


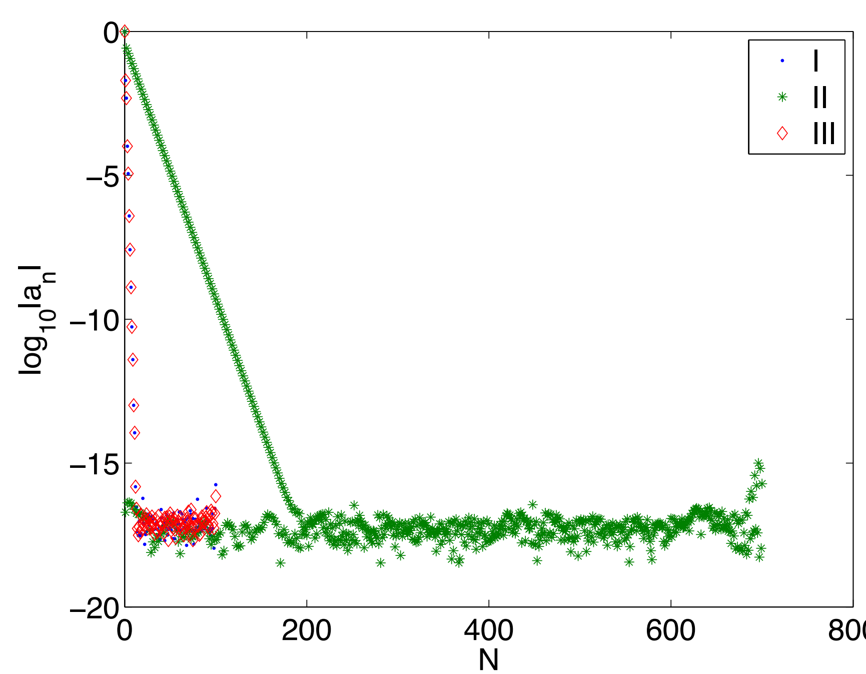
Fourth oder method

- $x_r = -x_l = 10$
- $N^I = N^{III} = 50,\ N^{II} = 700$
- $N_t = 1000, 2000$

$t = 0$ $t = 1$







Time integration, stiff systems

Method of lines

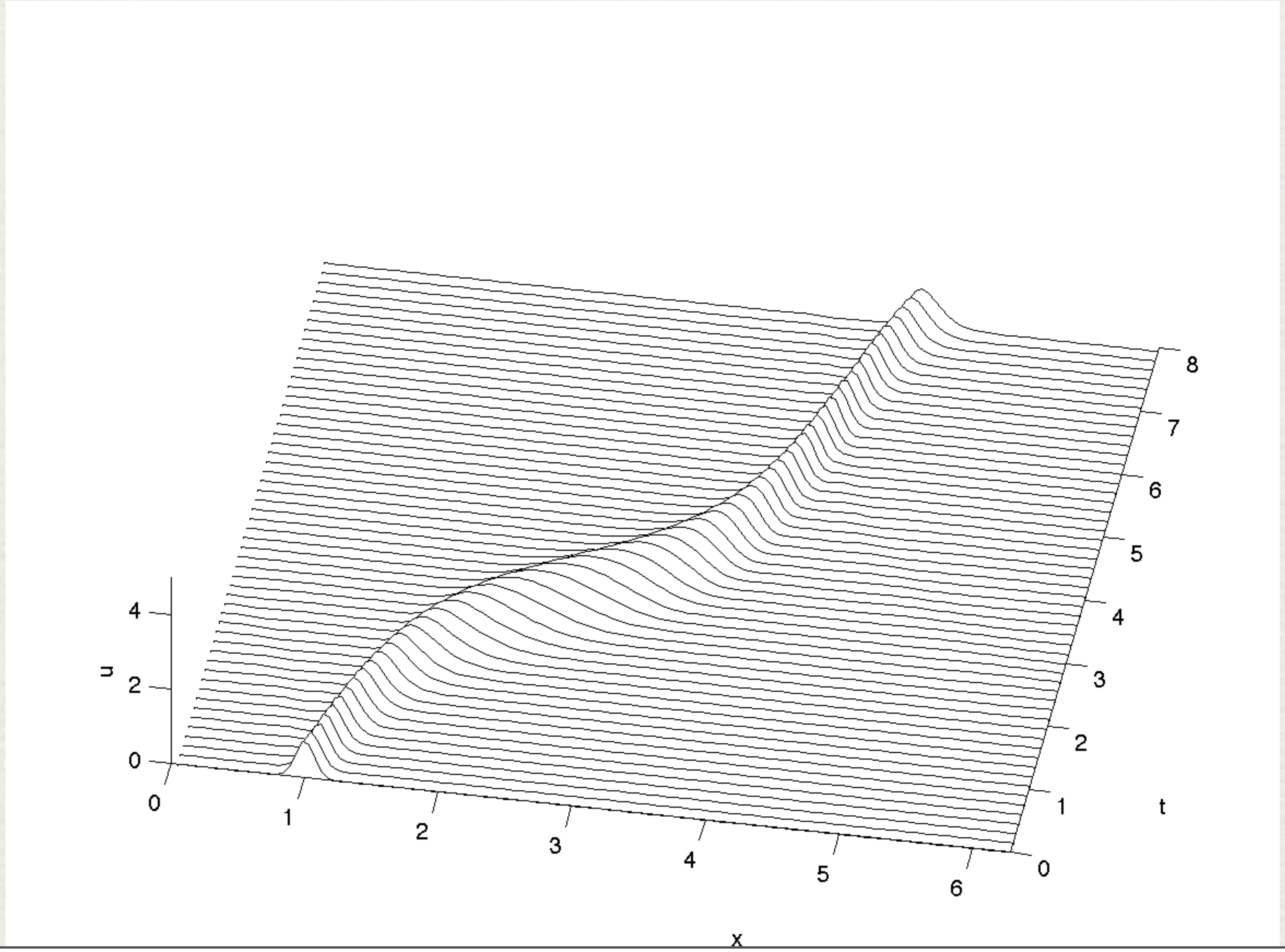
- spatial discretization of the PDE, system of ODEs
- finite differences for time integration (initial value problems, storage not of all computed time steps)
- Courant–Friedrichs–Lewy condition (time step delimited by studied velocities and spatial resolution)

Stability

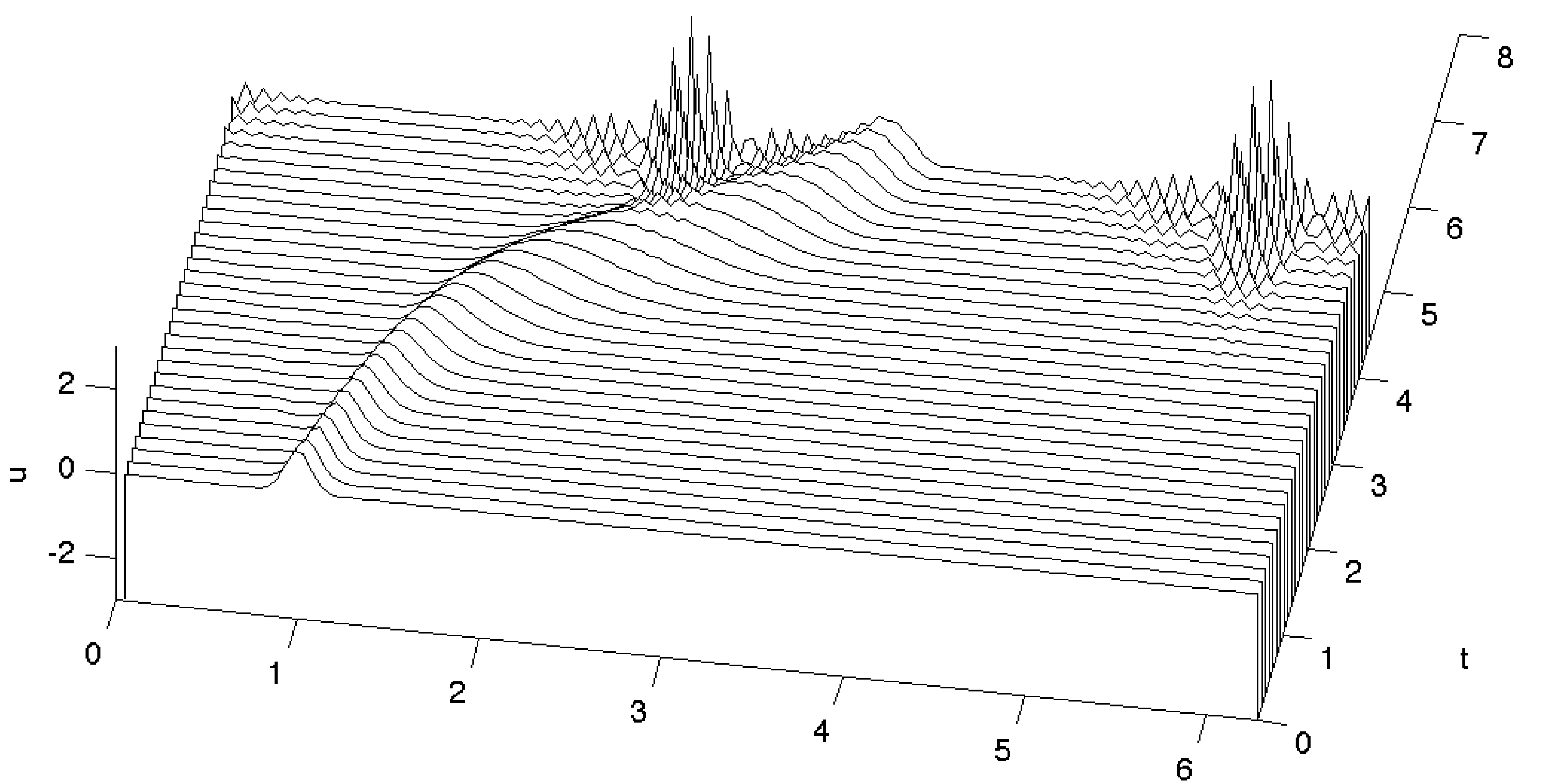
$$u_t + c(x)u_x = 0, \quad c(x) = \frac{1}{5} + \sin^2(x-1)$$

$$u(x,0) = \exp(-100(x-1)^2), \quad x \in [0, 2\pi]$$

$$u_t \sim \frac{u(t_{n+1}) - u(t_{n-1})}{2h}, \quad h = \frac{1.57}{N}$$



$$u_t \sim \frac{u(t_{n+1}) - u(t_{n-1})}{2h}, \quad h = \frac{1.9}{N}$$



Dahlquist

- model problem: $u_t = \lambda u$
- A-stable: numerical solution tends to zero for any initial data and any time step for negative real part of λ
- Dahlquist‘s second stability barrier: there is no A-stable linear multistep method of higher than second order

Stiff systems

- systems with vastly different scales
- explicit schemes inefficient for stability reasons
- implicit schemes computationally expensive for nonlinear equations if they have to be solved iteratively

Implicit schemes

- Crank-Nicolson (trapezoidal rule), for $y' = f(y,t)$:

$$y(t_{n+1}) = y(t_n) + \frac{h}{2}(f(y(t_n), t_n) + f(y(t_{n+1}), t_{n+1}))$$

second order method, A-stable

- implicit Runge-Kutta method of fourth order (two-stage Gauss scheme):

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i K_i,$$

$$K_i = f\left(t_n + c_i h,\, y_n + h\sum_{j=1}^{s} a_{ij} K_j\right),$$

$c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}$, $c_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}$, $a_{11} = a_{22} = 1/4$, $a_{12} = \frac{1}{4} - \frac{\sqrt{3}}{6}$, $a_{21} = \frac{1}{4} + \frac{\sqrt{3}}{6}$ and $b_1 = b_2 = 1/2$.

- Fourier space: equation of the form

$$U_t = cU + N[U]$$

here: U vector (1+1) or matrix (2+1), c array, $N[U]$ convolution, steep gradients: high frequency terms in c lead to large absolute values despite small ϵ

- exponential time differencing: time discretization and integration with integrating factor

$$U(t_n + h) = e^{ch}U(t_n) + \int_0^h d\tau e^{c(h-\tau)} N[U(t_n + \tau)]$$

fourth-order Runge-Kutta scheme (Cox-Matthews), coefficients via contour integrals (Kassam-Trefethen)

- integrating factor, fourth-order Runge-Kutta (e.g. Trefethen):

$$\left(e^{-ct}U\right)_t = e^{-ct}N[U]$$

- split equation of the form $u_t = (A+B)u$ (A, B self-adjoint operators) in

$$u_t = Au, \quad u_t = Bu$$

use approximations of the formula

$$\lim_{n\to\infty}\left(e^{-tA/n}e^{-tB/n}\right) = e^{(A+B)t}$$

for NLS

$$\psi_t = \frac{i\epsilon}{2}\psi_{xx}, \quad \psi_t = -\frac{i\rho}{\epsilon}|\psi|^2\psi$$

first PDE explicitly integrable in Fourier space, the second in physical space

- IMEX (implicit explicit)
idea: stable implicit scheme for the linear part, explicit scheme for the nonlinear part
Driscoll (*sliders*): stiffly stable 3rd order Runge-Kutta scheme for the high frequencies of the linear part, standard 4th order Runge-Kutta for the rest

Kadomtsev-Petviashvili equations

$$\partial_x \left(\partial_t u + 6u\partial_x u + \epsilon^2 \partial_{xxx} u\right) + \lambda \partial_{yy} u = 0, \ \lambda = \pm 1$$

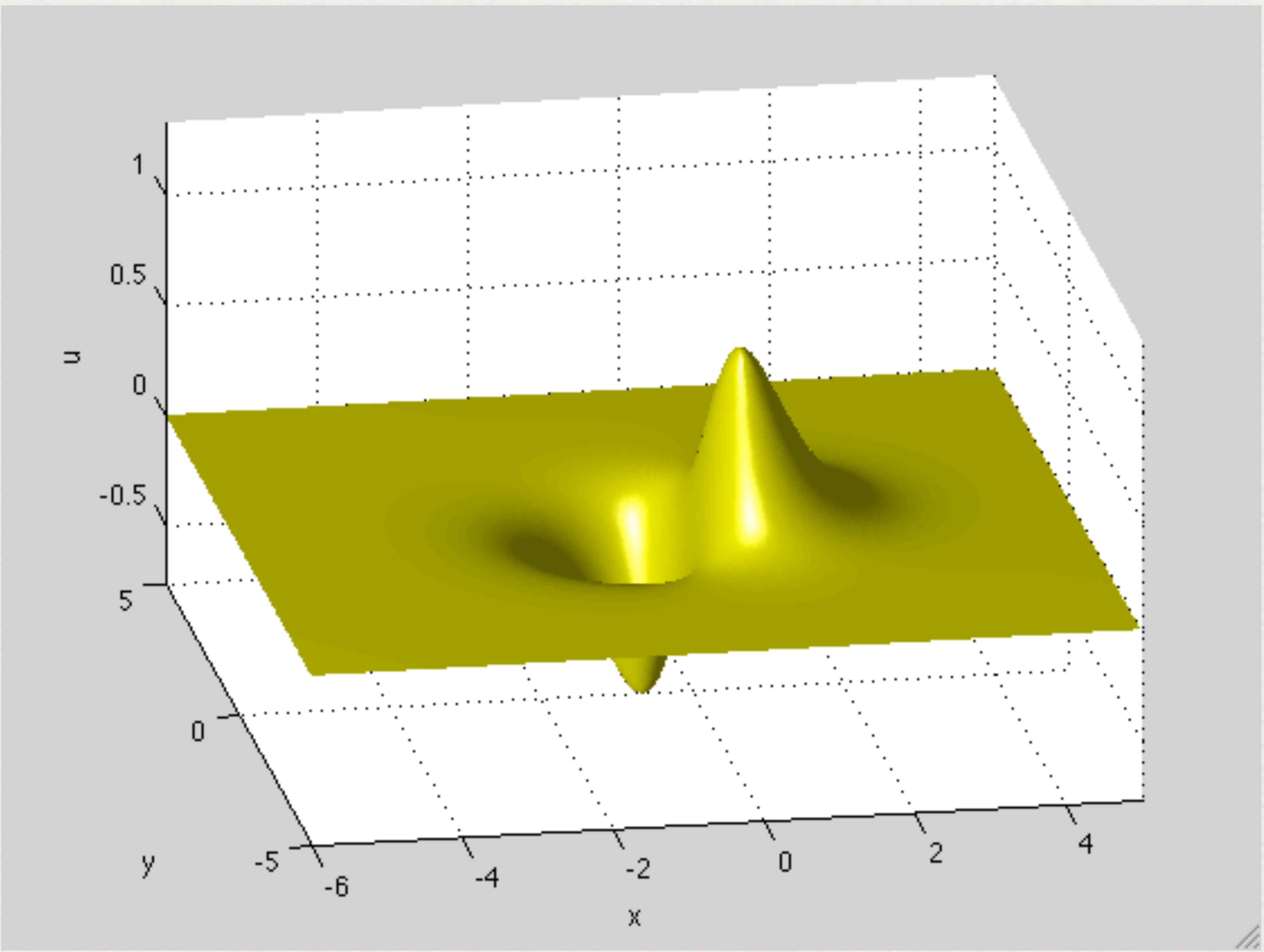
$$u_0(x, y) = -\partial_x \mathrm{sech}^2(R)$$

$$R = \sqrt{x^2 + y^2}$$

$$\epsilon = 0.1, \quad \lambda = -1$$

Kadomtsev-Petviashvili equations

$$\partial_x\left(\partial_t u + 6u\partial_x u + \epsilon^2\partial_{xxx}u\right) + \lambda\partial_{yy}u = 0,\ \lambda = \pm 1$$

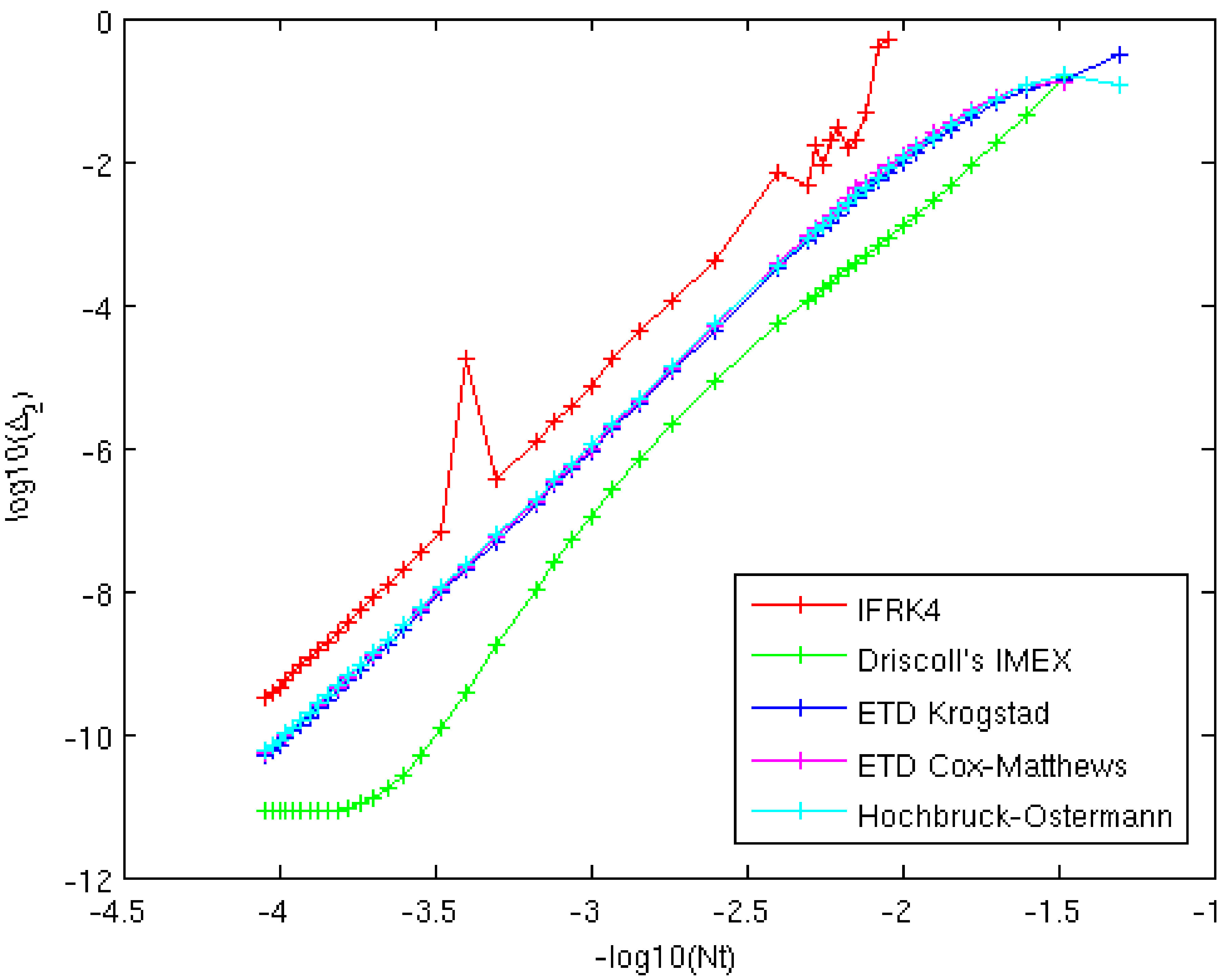


$$u_0(x,\,y) = -\partial_x \mathrm{sech}^2(R)$$

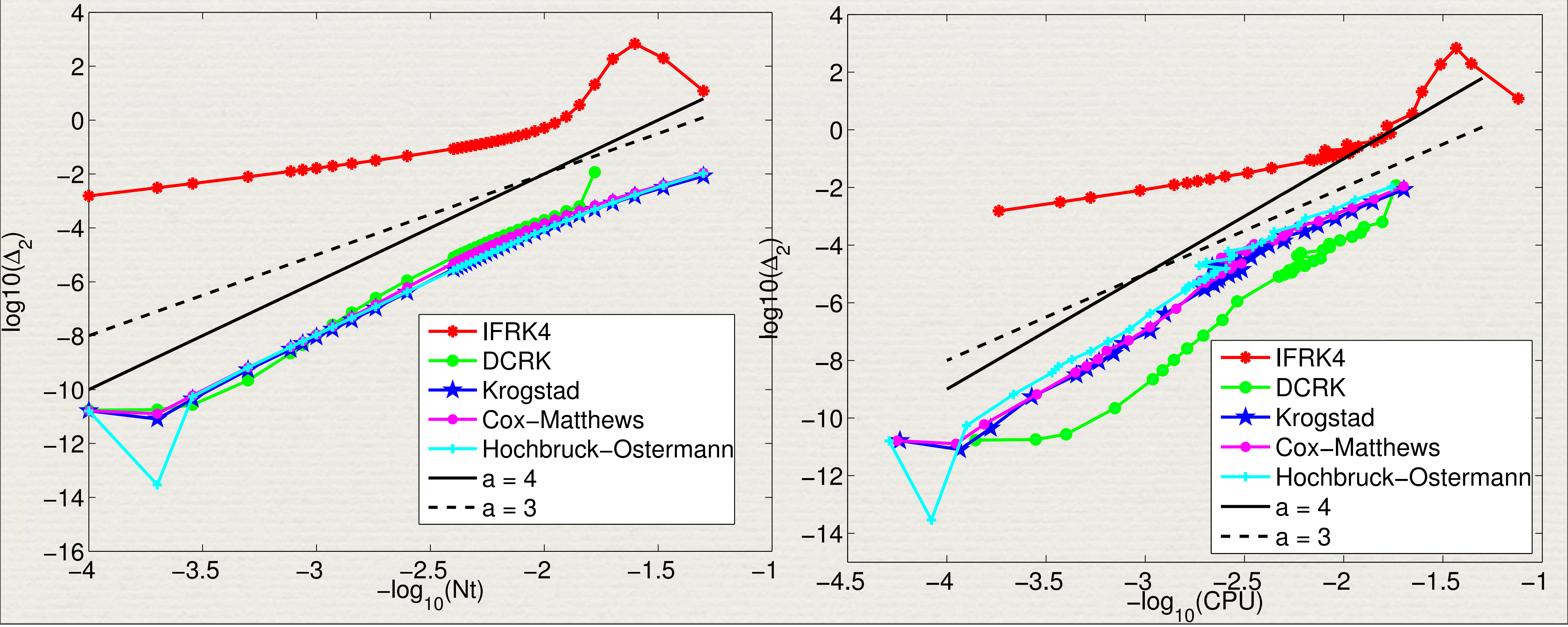
$$R = \sqrt{x^2+y^2}$$

$$\epsilon = 0.1, \quad \lambda = -1$$

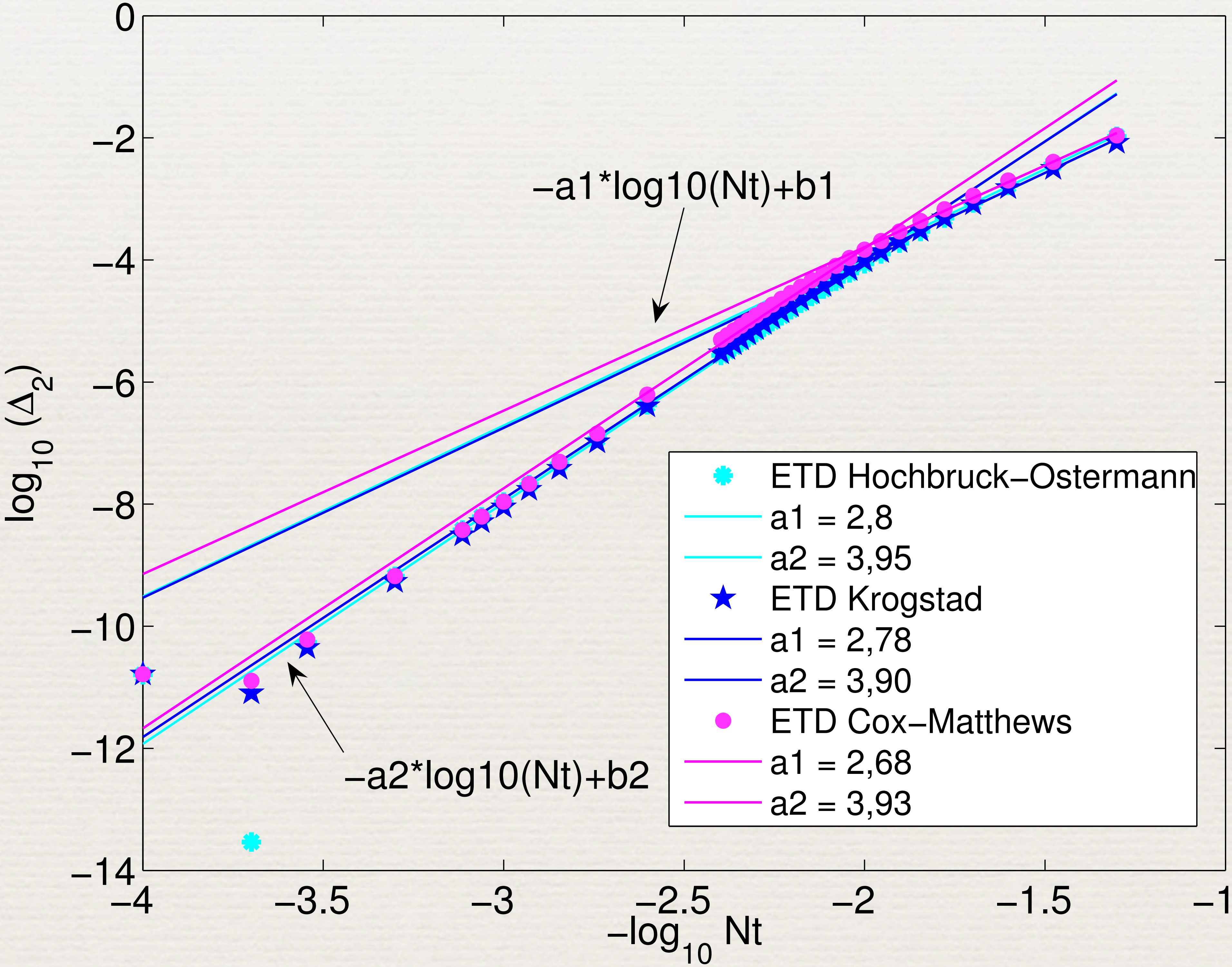
Zaitsev solution, difference between numerical and exact solution



Performance



Order reduction



Davey-Stewartson equation

C. Klein and K. Roidot, *Numerical Study of the semiclassical limit of the Davey-Stewartson II equations*, Nonlinearity 27, 2177-2214 (2014).

- $$\begin{aligned} i\epsilon u_t + \epsilon^2 u_{xx} - \alpha\epsilon^2 u_{yy} + 2\rho\left(\Phi + |u|^2\right)u &= 0 \\ \Phi_{xx} + \alpha\Phi_{yy} + 2\,|u|^2_{xx} &= 0 \end{aligned}$$

- integrable cases: $\alpha = \pm 1$, $\rho = \pm 1$
  - DS I, $\alpha = -1$
  - DS II, hyperbolic-elliptic, $\alpha = 1$

- y-independent potential plus boundary condition at infinity: reduction to NLS

- first numerical studies: White-Weideman (1994), Besse, Mauser, Stimming (2004), McConnell, Fokas, Pelloni (2005)

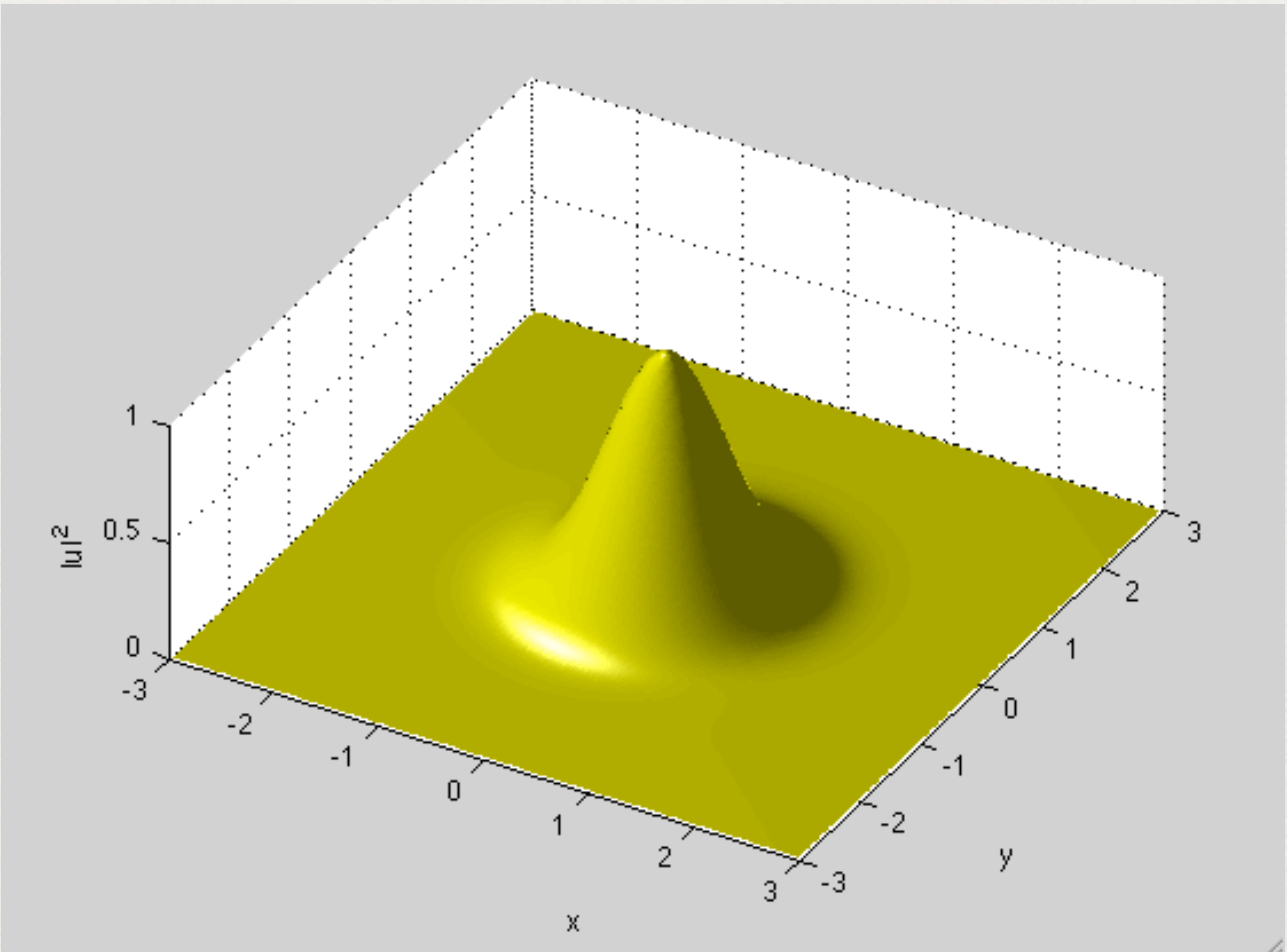
Defocusing DS II

$u_0 = \exp(-x^2 - y^2)$

$\epsilon = 0.1$

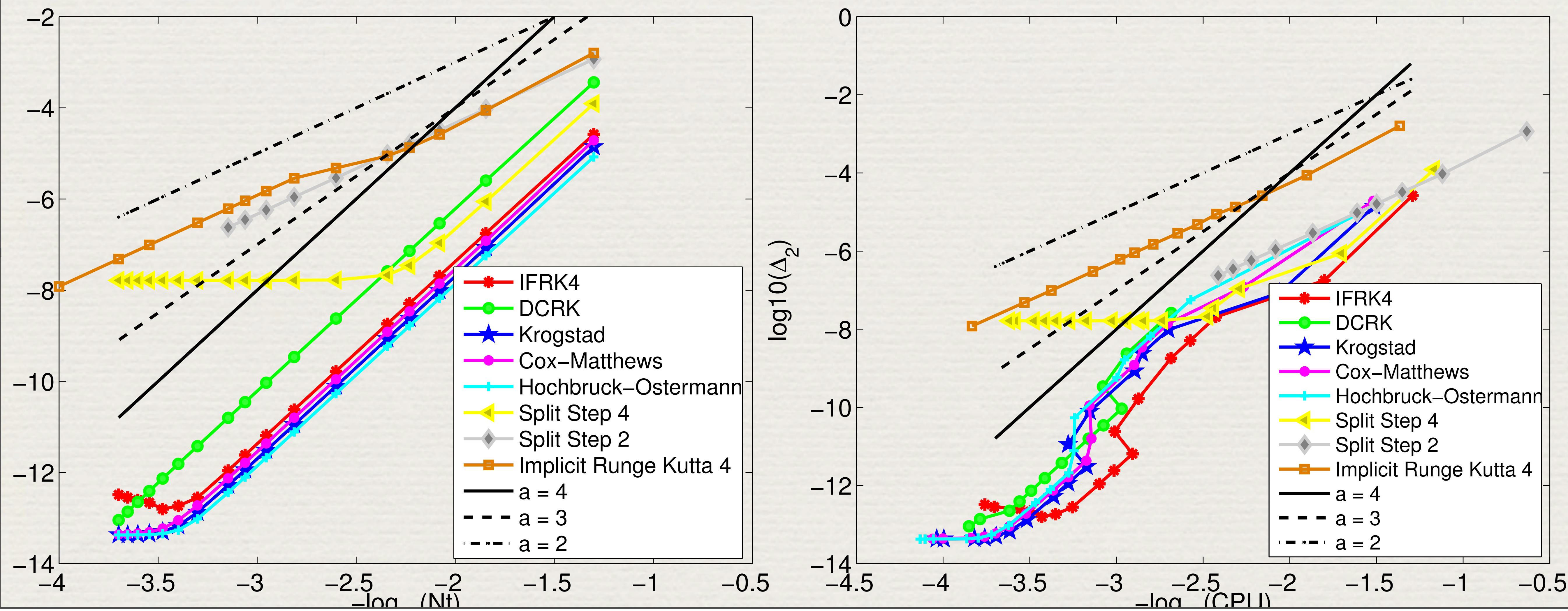
Defocusing DS II

$u_0 = \exp(-x^2 - y^2)$



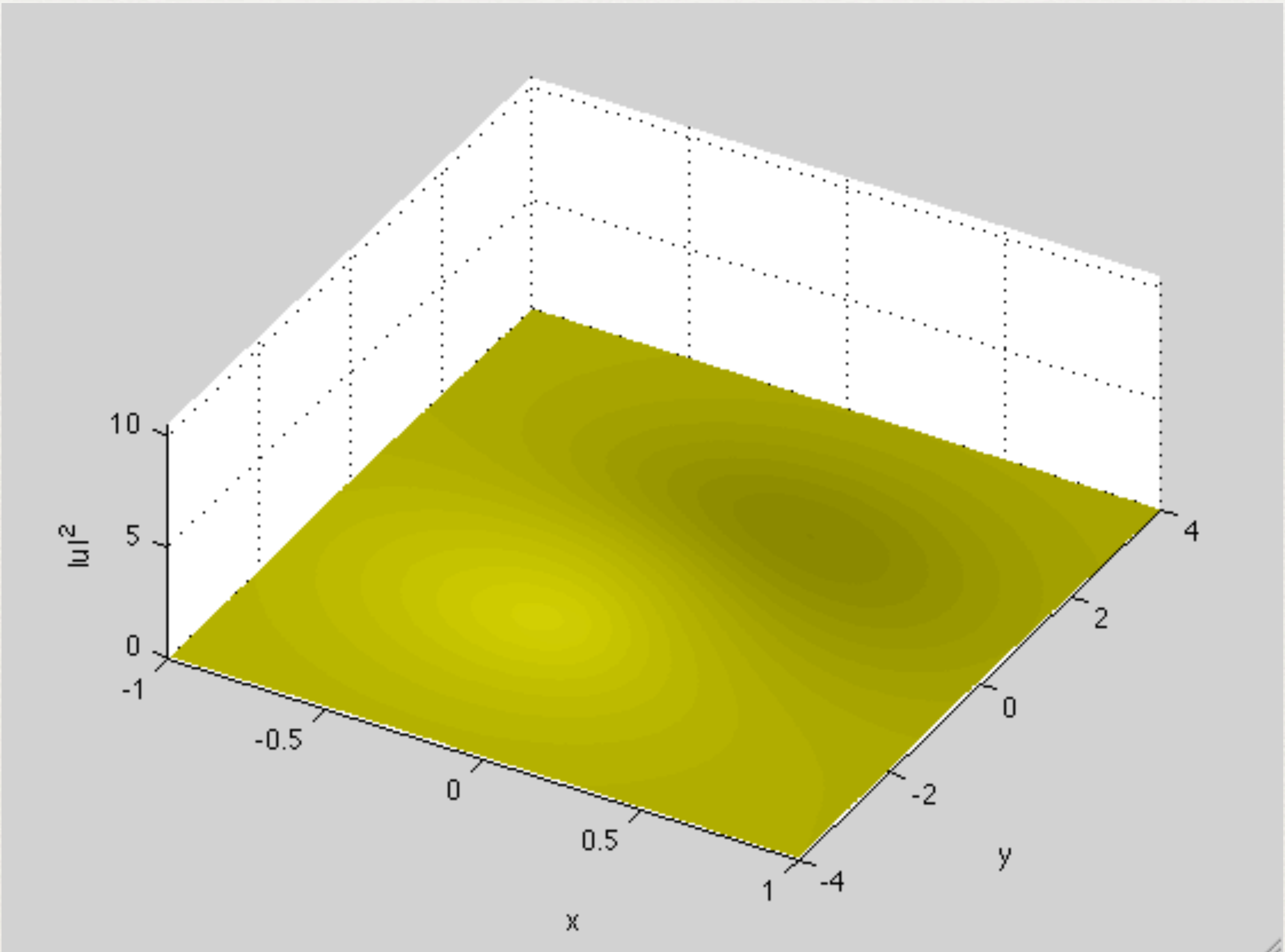
$\epsilon = 0.1$

Performance



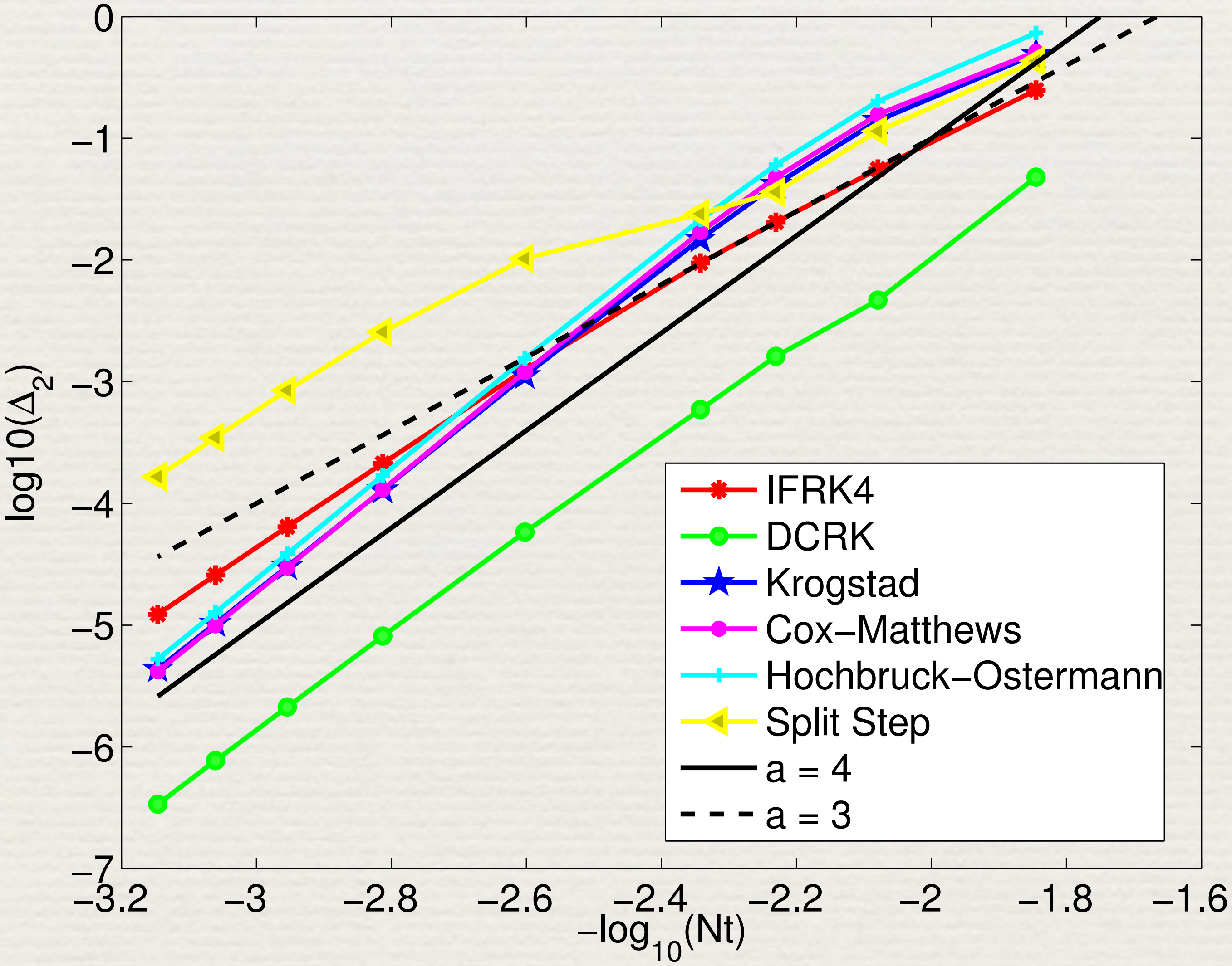
Focusing DS

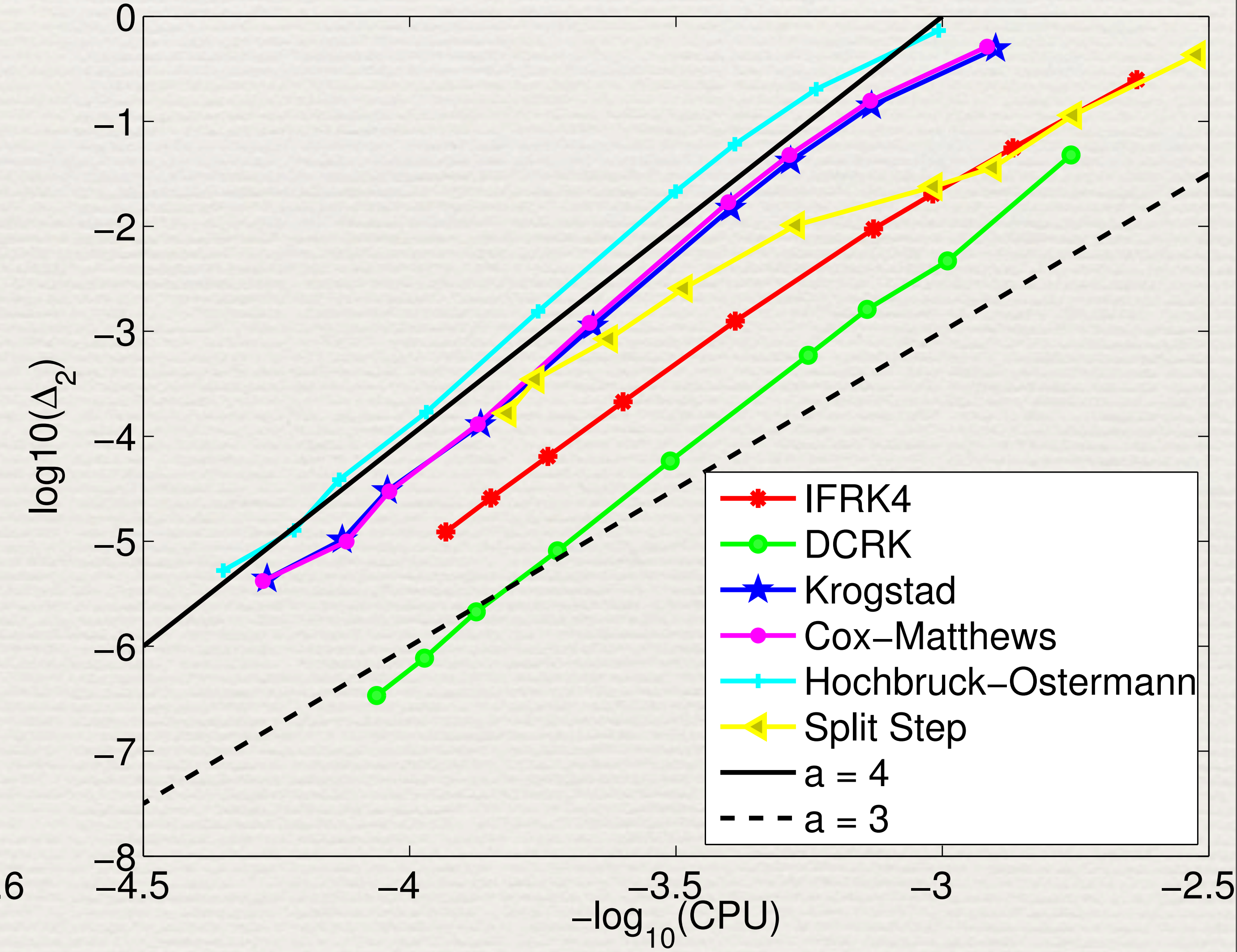
$u_0 = \exp(-x^2 - 0.1y^2)$



$\epsilon = 0.1$

Performance





Breather

$$\psi = 4\exp(it/2)\frac{\cosh(3x)+3\exp(4it)\cosh(x)}{\cosh(4x)+4\cosh(2x)+3\cos(4t)}$$

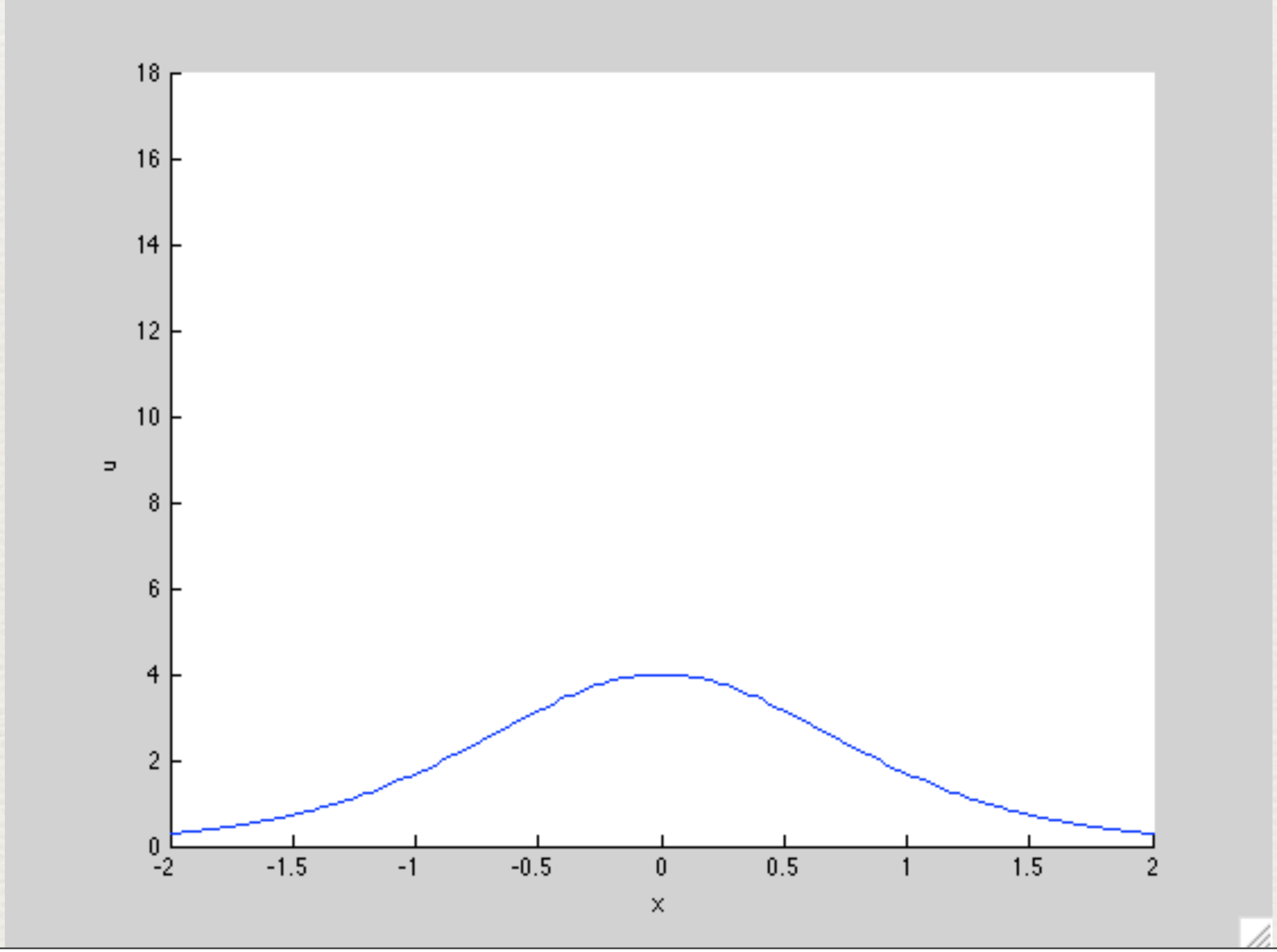
$$\psi_0 = 2\text{sech}x$$

$$u = |\psi|^2$$

Breather

$$\psi = 4\exp(it/2)\frac{\cosh(3x)+3\exp(4it)\cosh(x)}{\cosh(4x)+4\cosh(2x)+3\cos(4t)}$$

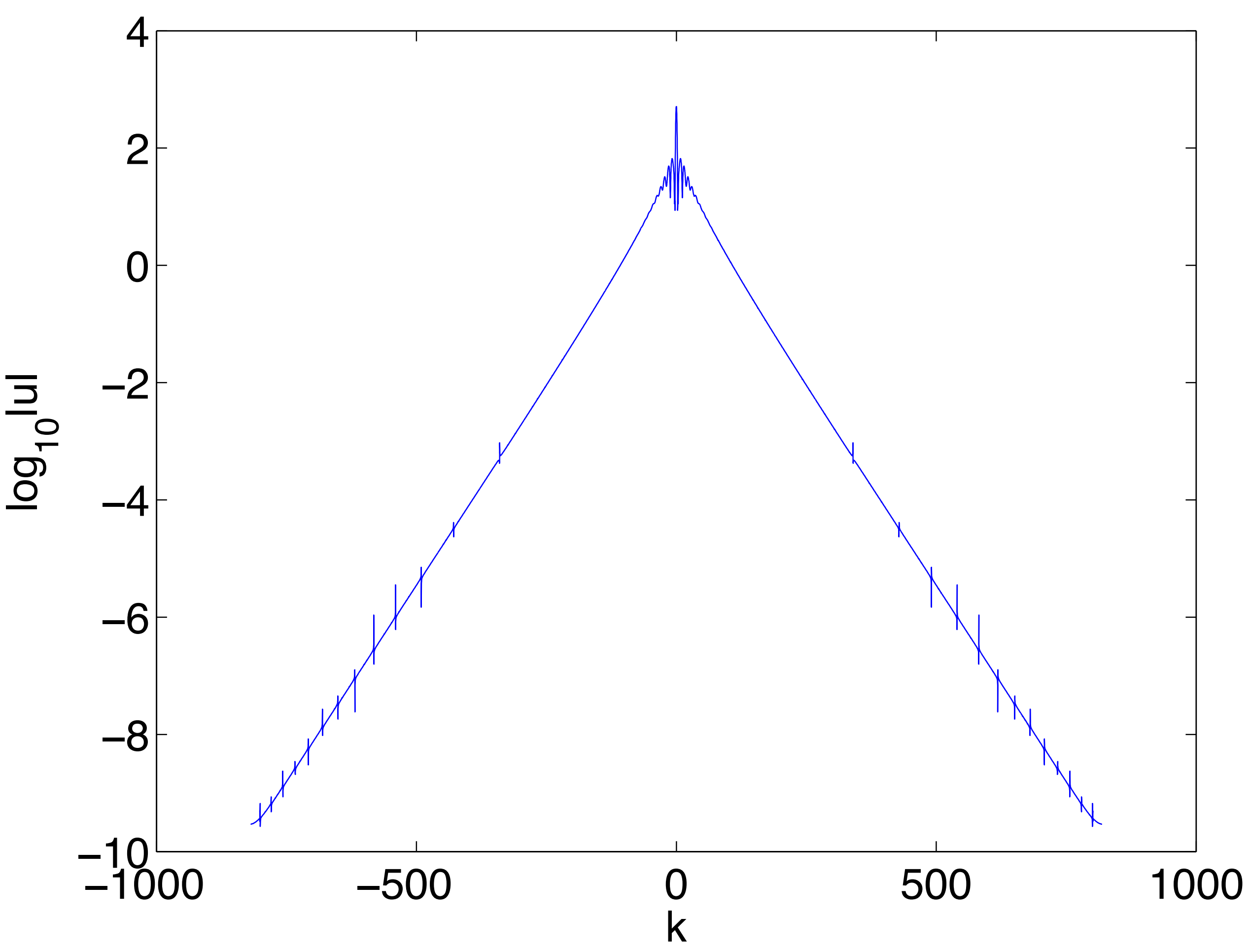
$$\psi_0 = 2\text{sech}x$$

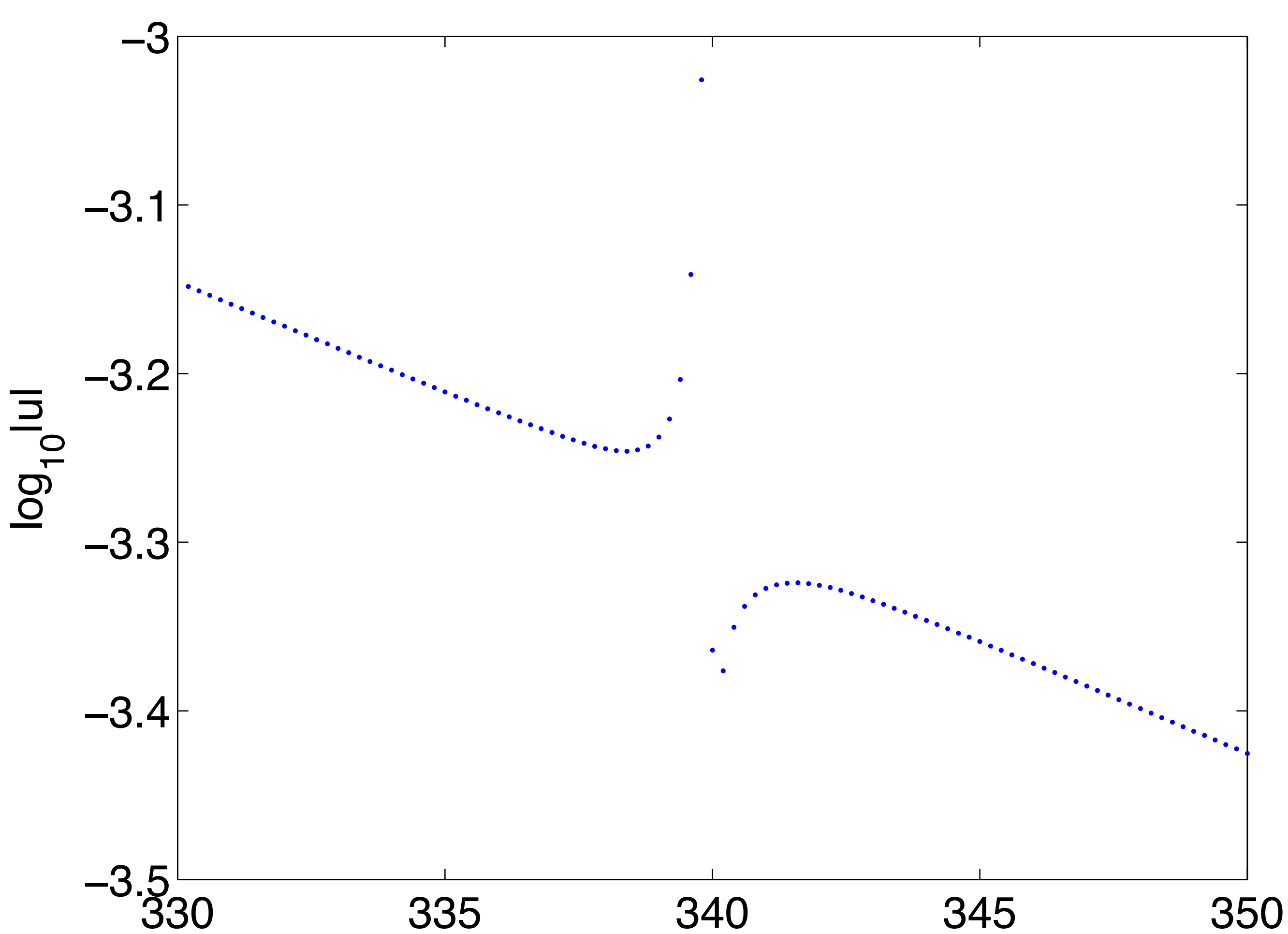


$$u = |\psi|^2$$

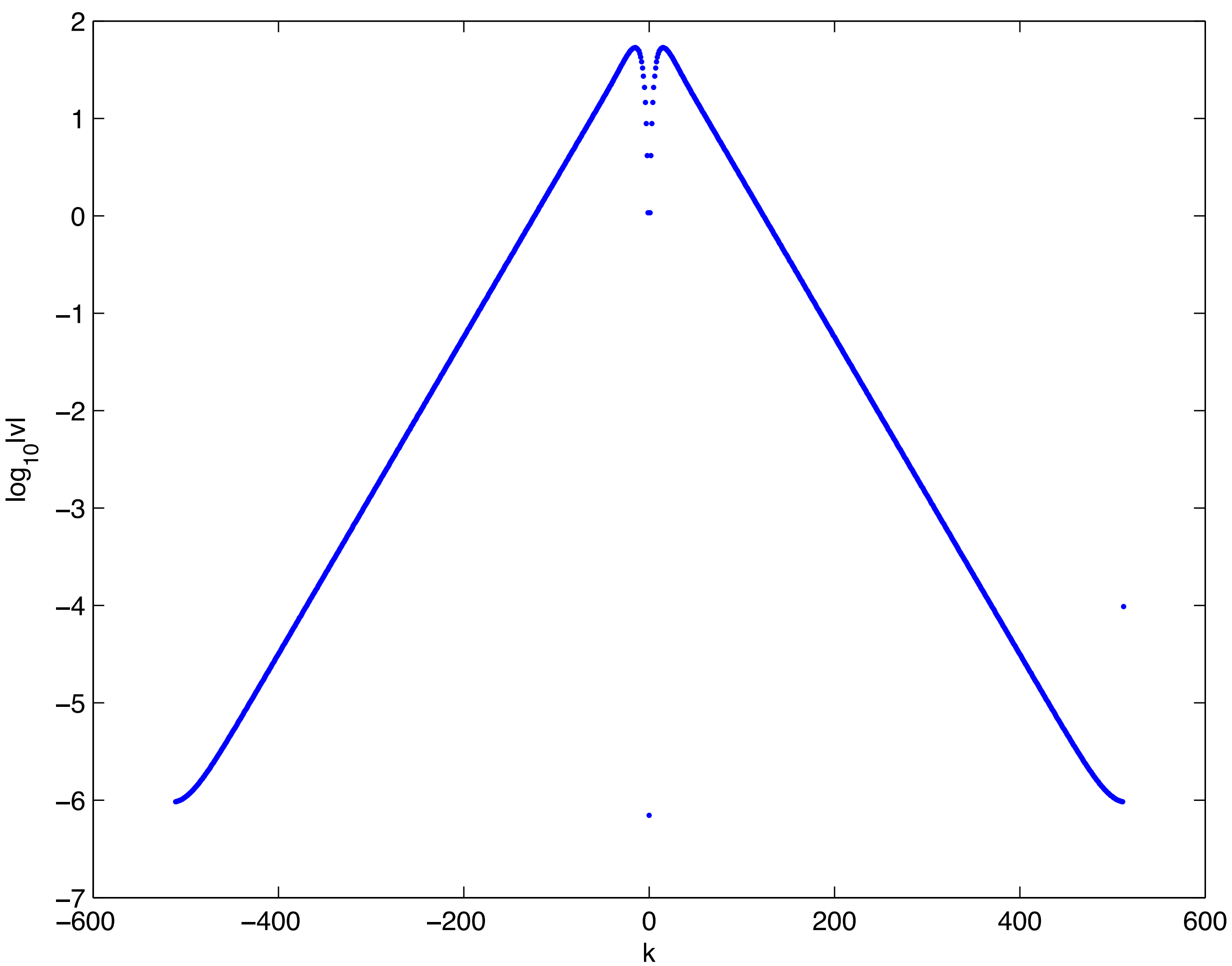
Instabilities of exponential integrators, gKdV

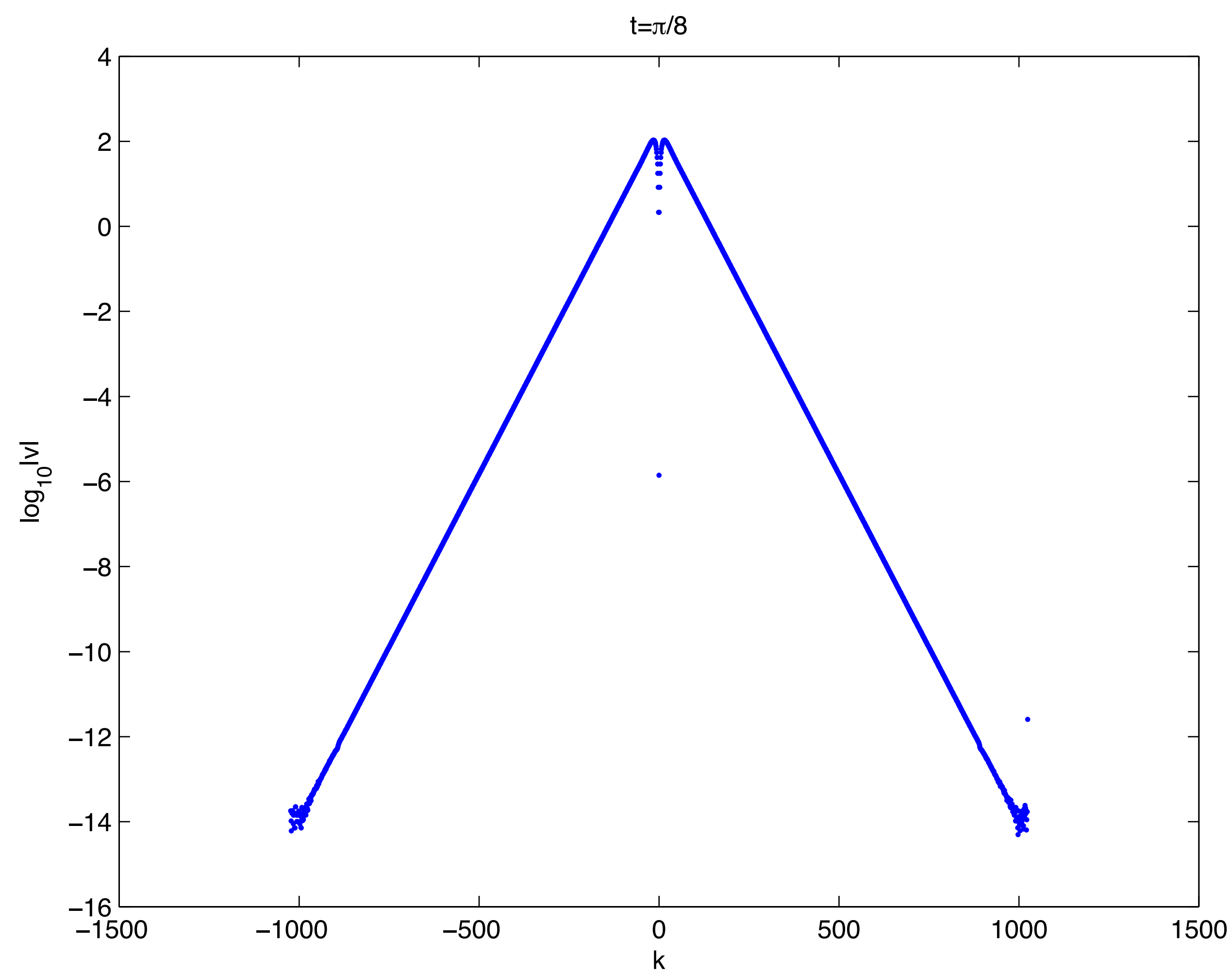
$$u_0(x) = \text{sech}^2 x,\ n = 4,\ t = 0.318 \gg t_c,\ \epsilon = 0.1$$



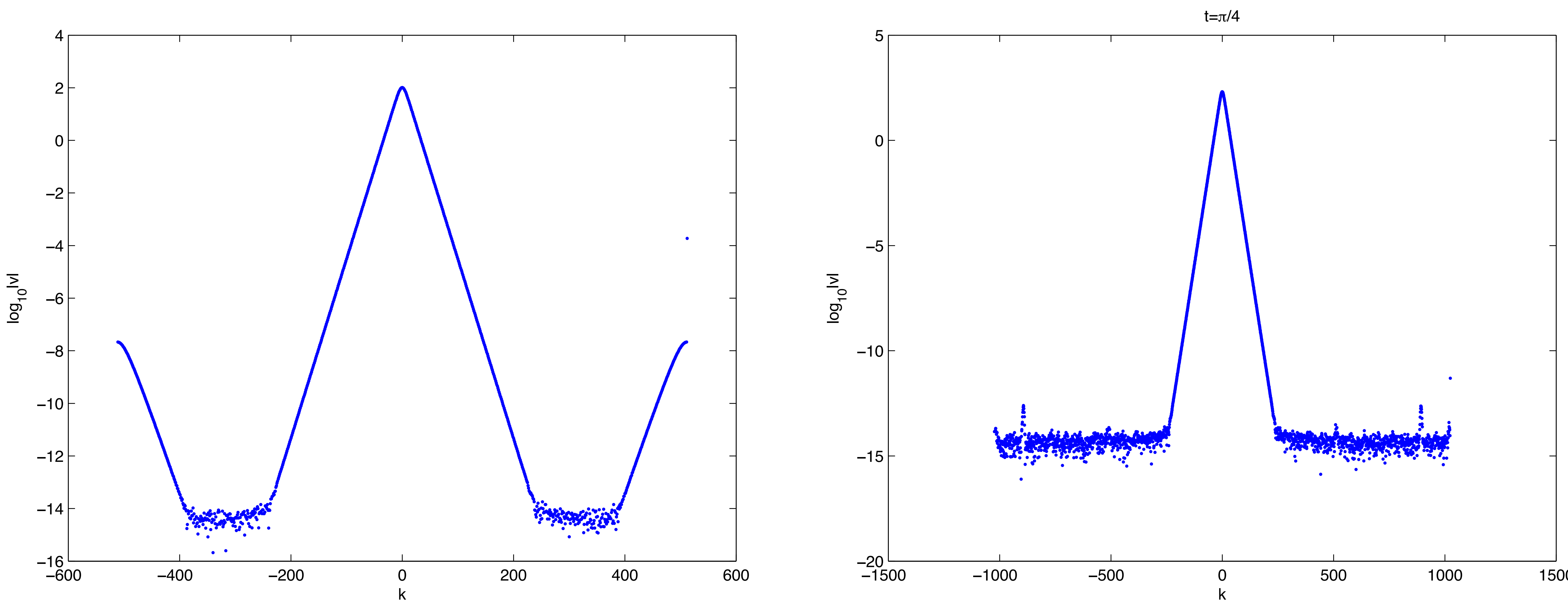


breather, $t = \pi/8$





Modulational instability: breather, t = π/4



References

[1] K. A. Bagrinovskii and S. K. Godunov, *Difference schemes for multi-dimensional problems*, Dokl. Acad. Nauk, 115 (1957), pp. 431—433.

[2] S. M. Cox and P. C. Matthews., *Exponential time differencing for stiff systems*, J. Comput. Phys., 176(2) (2002), pp. 430—455.

[3] T. A. Driscoll, *A composite Runge–Kutta method for the spectral solution of semilinear PDEs*, J. Comput. Phys., 182 (2002), pp. 357-–367.

[4] M. Hochbruck and A. Ostermann, *Exponential Runge-Kutta methods for semilinear parabolic problems*, SIAM J. Numer. Anal., 43(3) (2005), pp. 1069—1090.

[5] M. Hochbruck, A. Ostermann Exponential integrators. Acta Numerica 19, 209-286 (2010)

[6] A.-K. Kassam and L. N. Trefethen, *Fourth-order time-stepping for stiff PDEs*, SIAM J. Sci. Comput., 26 (2005), pp. 1214—1233.

[7] C. Klein, 'Fourth order time-stepping for low dispersion Korteweg-de Vries and nonlinear Schrödinger equation', ETNA Vol. 29 116-135 (2008).

[8] C. Klein and K. Roidot, 'Fourth order time-stepping for Kadomtsev-Petviashvili and Davey-Stewartson equations', SIAM Journal on Scientific Computing Vol. 33, No. 6, DOI: 10.1137/100816663 (2011).

Examples

C. Klein,
IMB (Dijon),
with
R. Peter
J.-C. Saut (Orsay)

Dynamic rescaling

- coordinate change ($L = L(\tau)$, $x_m = x_m(\tau)$)

$$\xi = \frac{x - x_m}{L}, \quad \frac{d\tau}{dt} = \frac{1}{L^3}, \quad U = L^{2/n} u$$

- rescaled equation, $a = (\ln L)_\tau, \quad v = x_{m,\tau}/L$

$$U_\tau - a\left(\frac{2}{n}U + \xi U_\xi\right) - vU_\xi + U^n U_\xi + \epsilon^2 U_{\xi\xi\xi} = 0$$

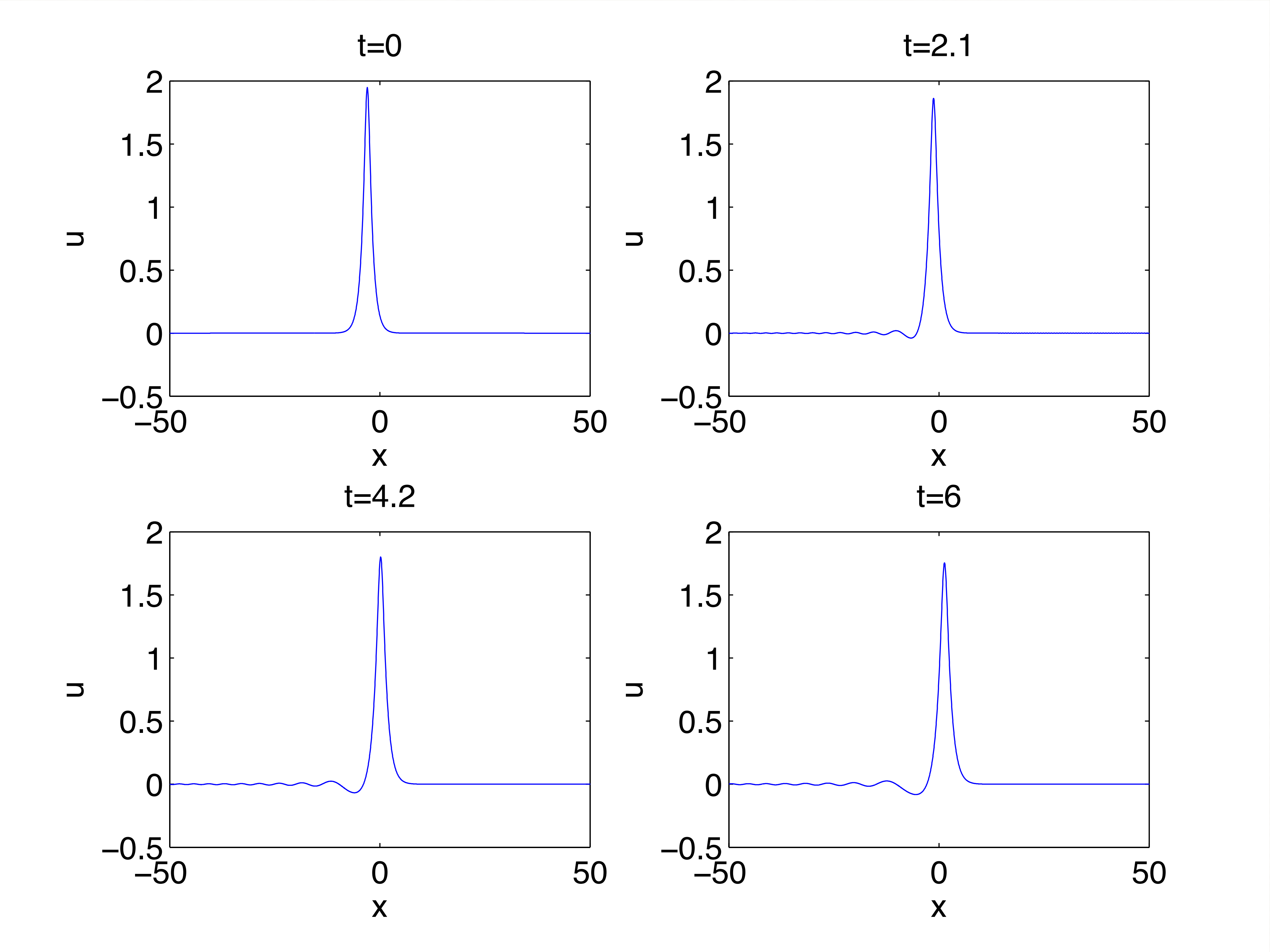
- blow-up: $L \to 0$, $a_\infty = const$, $U_\tau \to 0$

$$-a_\infty\left(\frac{2}{n}U_\infty + \xi U_{\infty,\xi}\right) - v_\infty U_{\infty,\xi} + U_\infty^n U_{\infty,\xi} + \epsilon^2 U_{\infty,\xi\xi\xi} = 0$$

- choice of L for numerical implementation: keep certain norms constant ($n = 4$: $||u||_2$ invariant under the rescaling, keep $||u_x||_2$ constant)
- Martel, Merle, Raphaël for $n = 4$, perturbed soliton: $a_\infty = 0$, therefore rescaled soliton, $x_m \to \infty$;
  condition for blow-up: energy smaller than soliton energy, mass larger
- no theory for the supercritical case $n > 4$, numerical experiments by Bona et al., Dix-McKinney

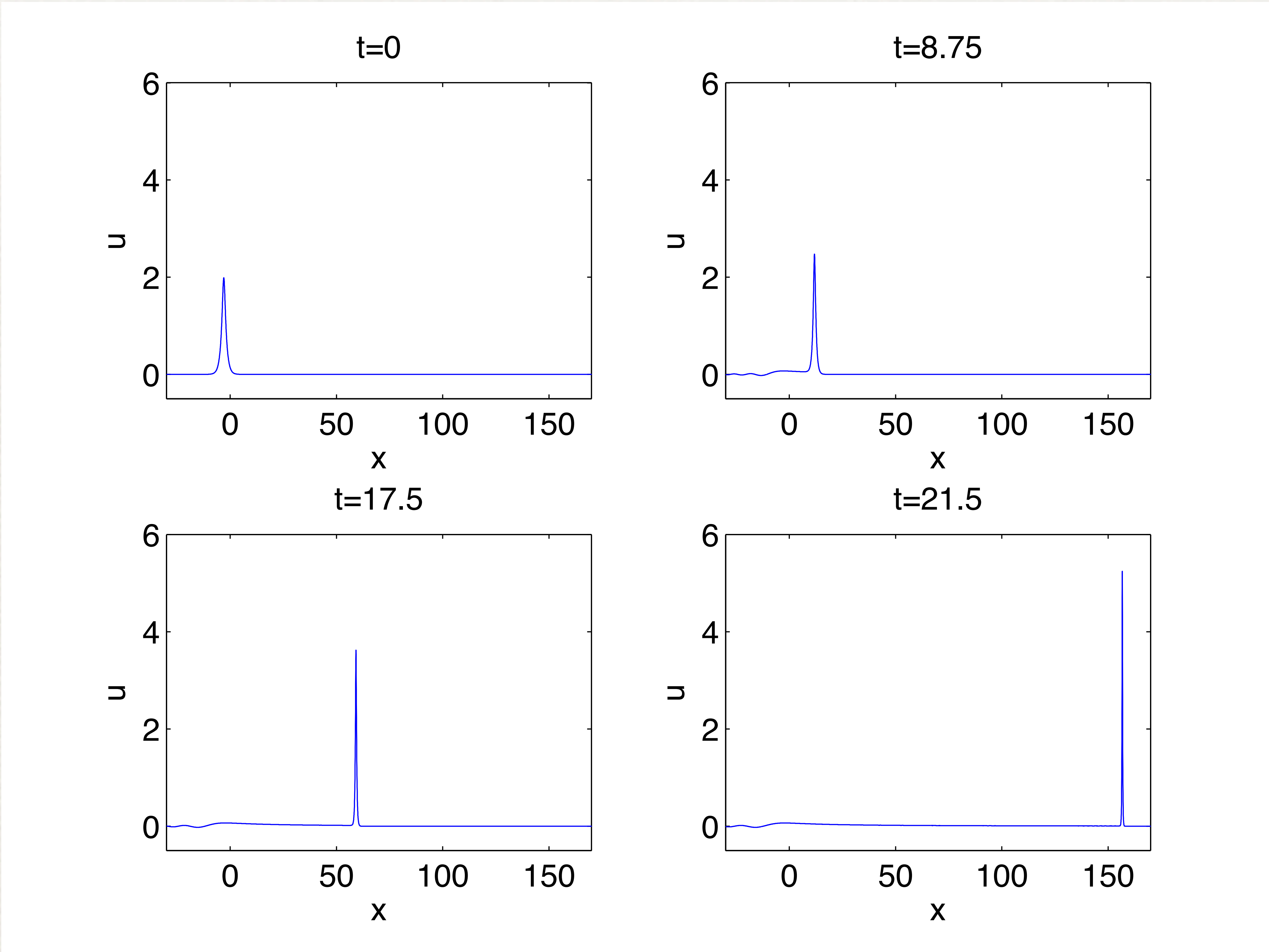
Perturbed soliton, n=4

$$u_0 = 0.99 u_{sol}$$



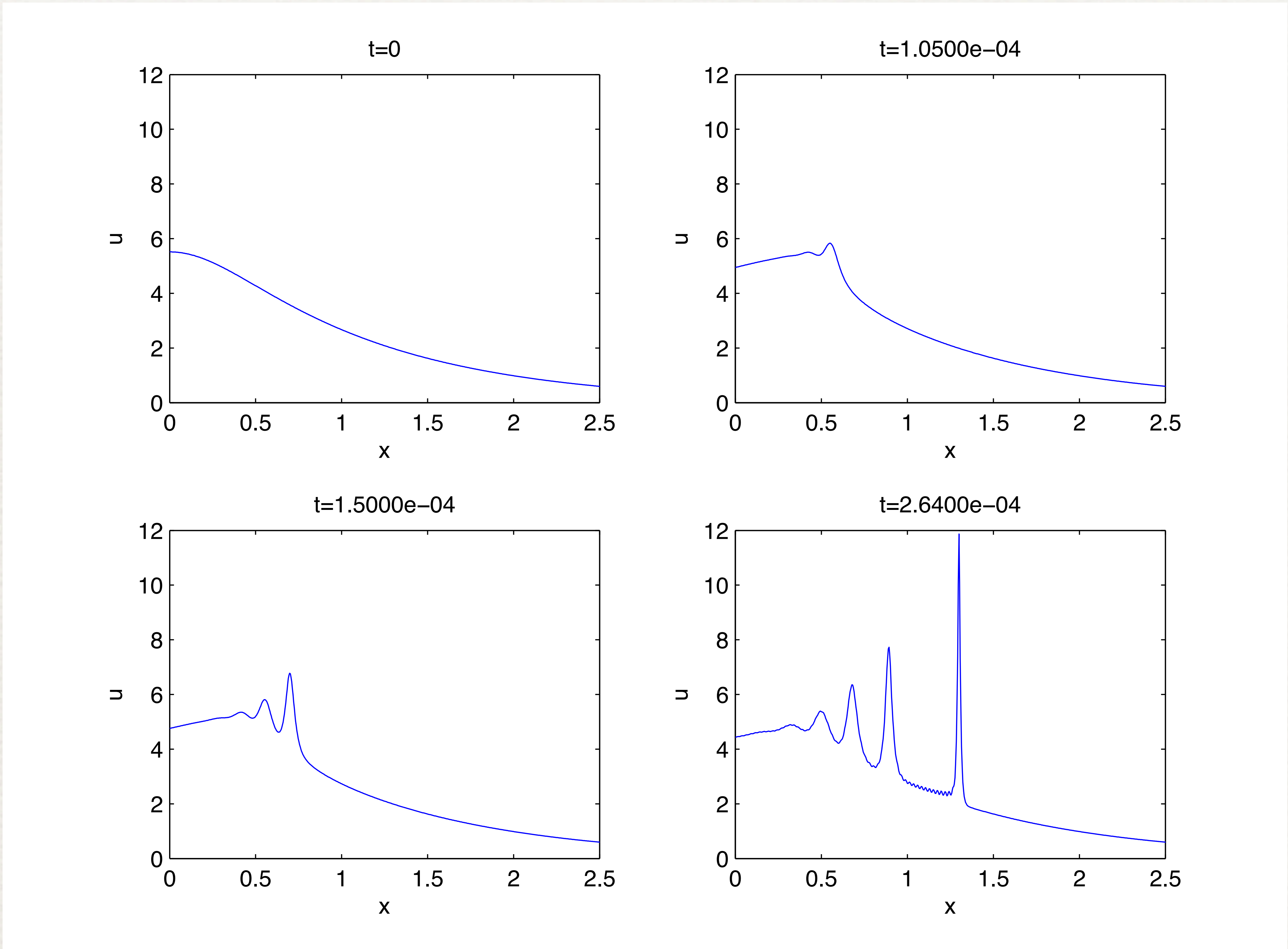
Perturbed soliton, n=4

$$u_0 = 1.01 u_{sol}$$



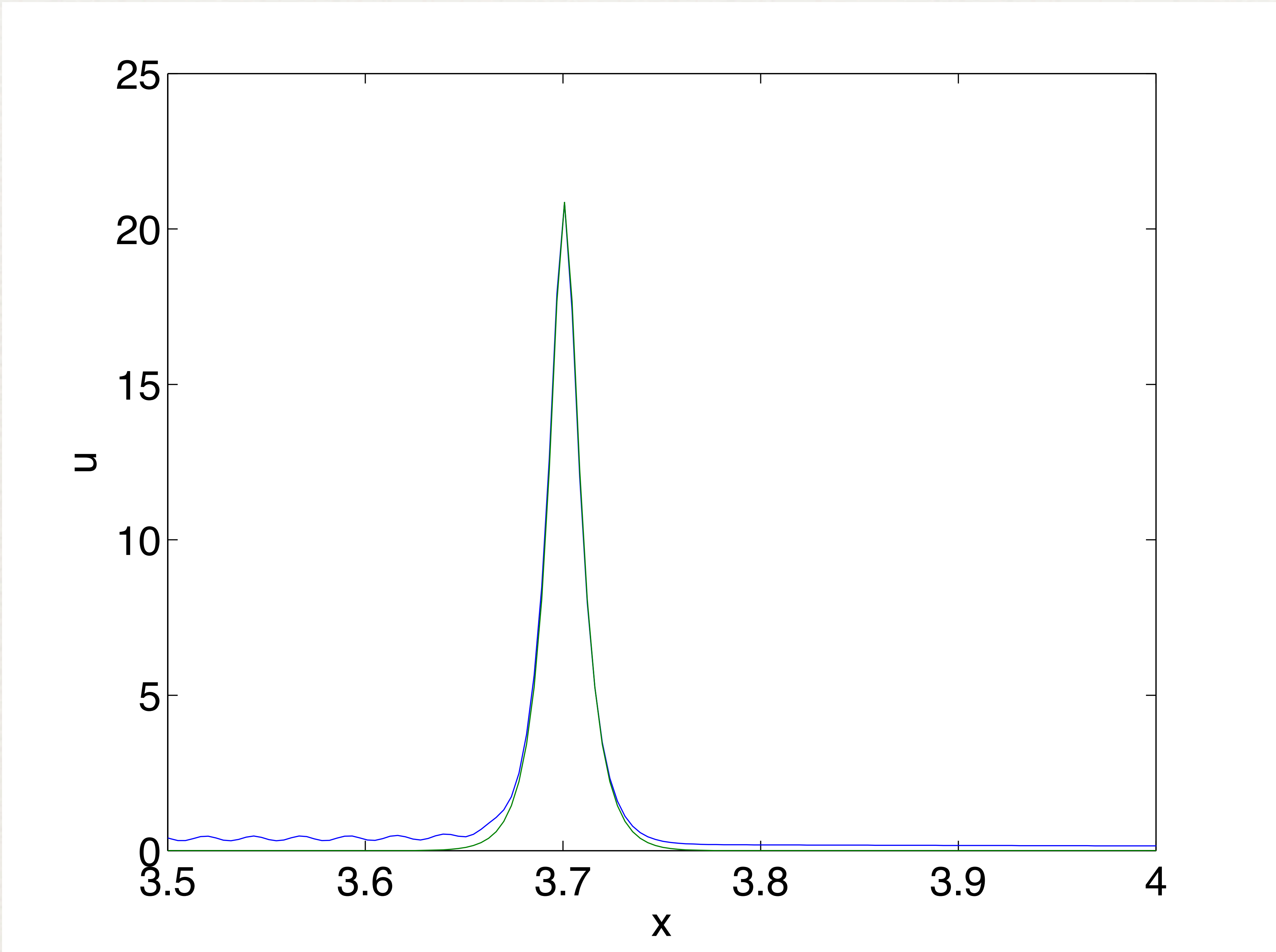
General localized initial, n=4

$$u_0 = 3u_{sol}$$



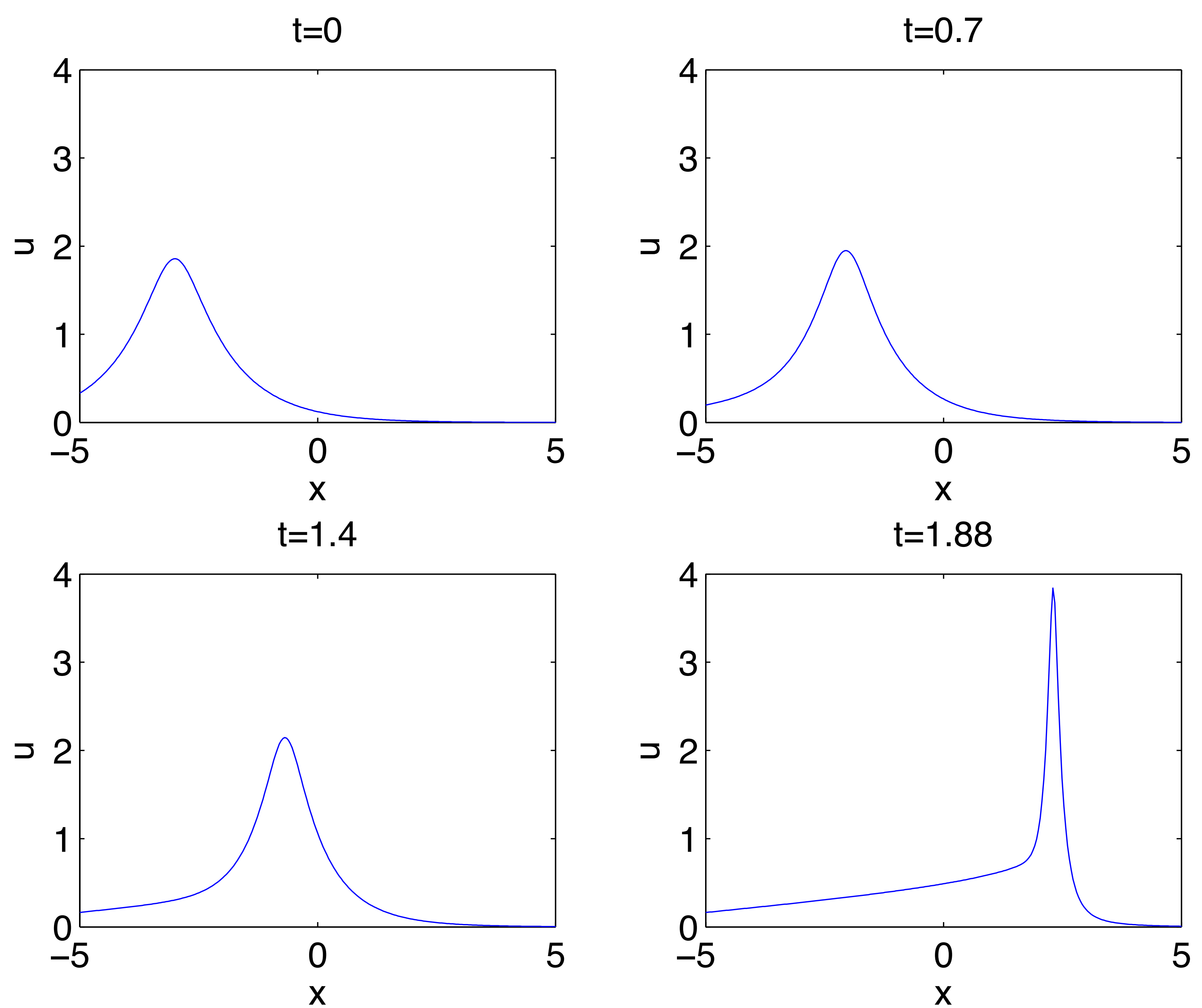
Fitting to rescaled soliton

$$u_0 = 3u_{sol}$$



Supercritical case, n=5

$$u_0 = 1.01 u_{sol}$$



Fractional KdV equations

- in contrast to gKdV, lower the dispersion: fractionary KdV (fKdV) equation

$$u_t + uu_x - D^\alpha u_x = 0, \quad \widehat{D^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi)$$

$\alpha = 2$: KdV, $\alpha = 1$: Benjamin-Ono

- Solitons $u(x,t) = Q_c(x - ct)$ (Linares, Pilod, Saut 2013)

$$D^\alpha Q_c + cQ_c - \frac{1}{2}Q_c^2 = 0,$$

solitons exist for $\alpha \geq 1/3$, not for $\alpha < 1/3$, algebraic decay for $x \to \infty$

- no blow-up expected for $\alpha > 1/2$

Dynamic rescaling

- coordinate change

$$y = \frac{x - x_m}{L}, \quad \frac{d\tau}{dt} = \frac{1}{L^{1+\alpha}}, \quad U = L^\alpha u$$

$||u||_2$ invariant for $\alpha = 1/2$, energy invariant for $\alpha = 1/3$

- rescaled equation

$$U_\tau - (\ln L)_\tau (\alpha U + yU_y) - \frac{x_{m,\tau}}{L} U_y + UU_y - D_y^\alpha U = 0$$

- blow-up

$$a_\infty \left(\alpha U^\infty + yU_y^\infty\right) - v_\infty U_y^\infty + U^\infty U_y^\infty - D_y^\alpha U^\infty = 0$$

$a_\infty = 0$, rescaled soliton

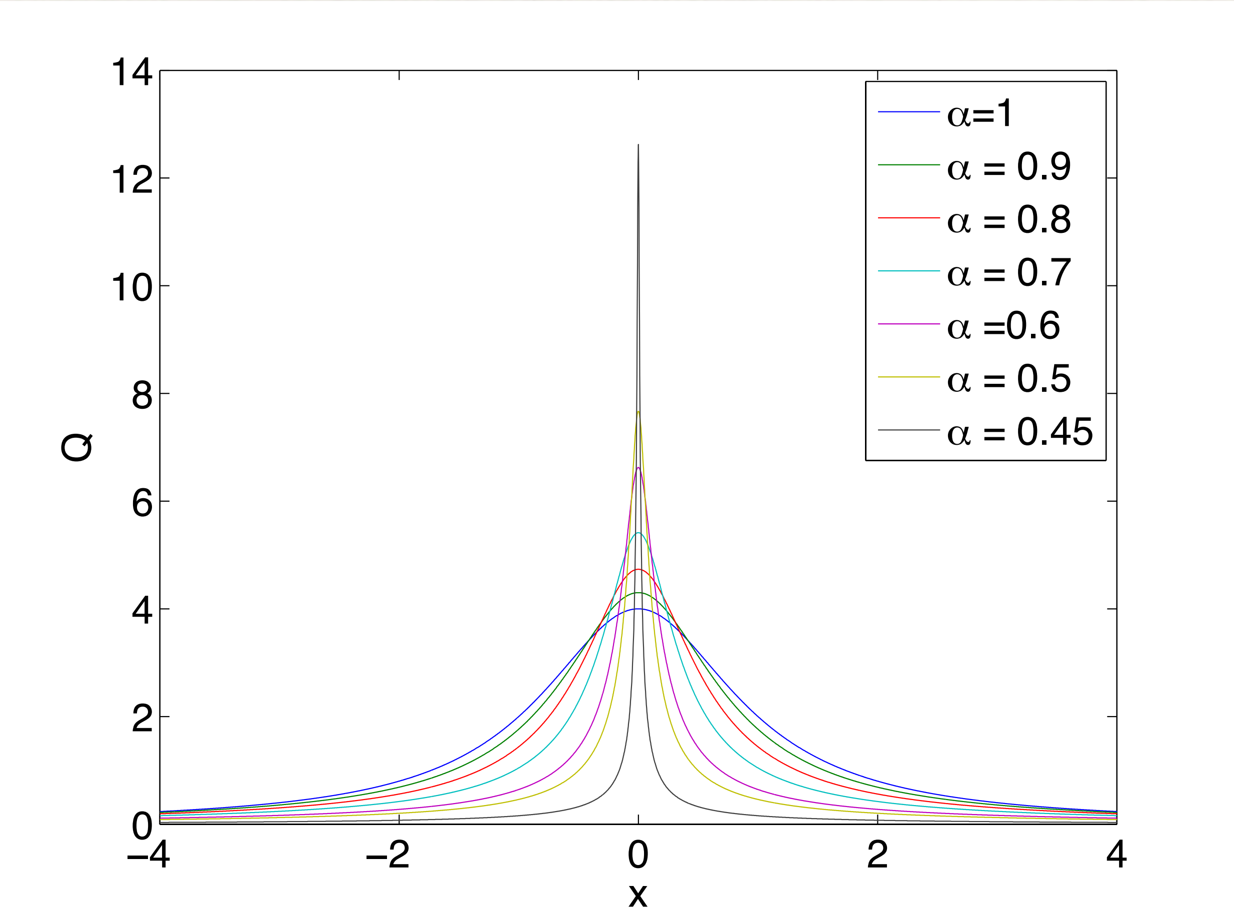
- slow decrease of the amplitude of the oscillations leads to numerical instabilities for the rescaled equation, therefore direct integration, scaling via norms of the solution

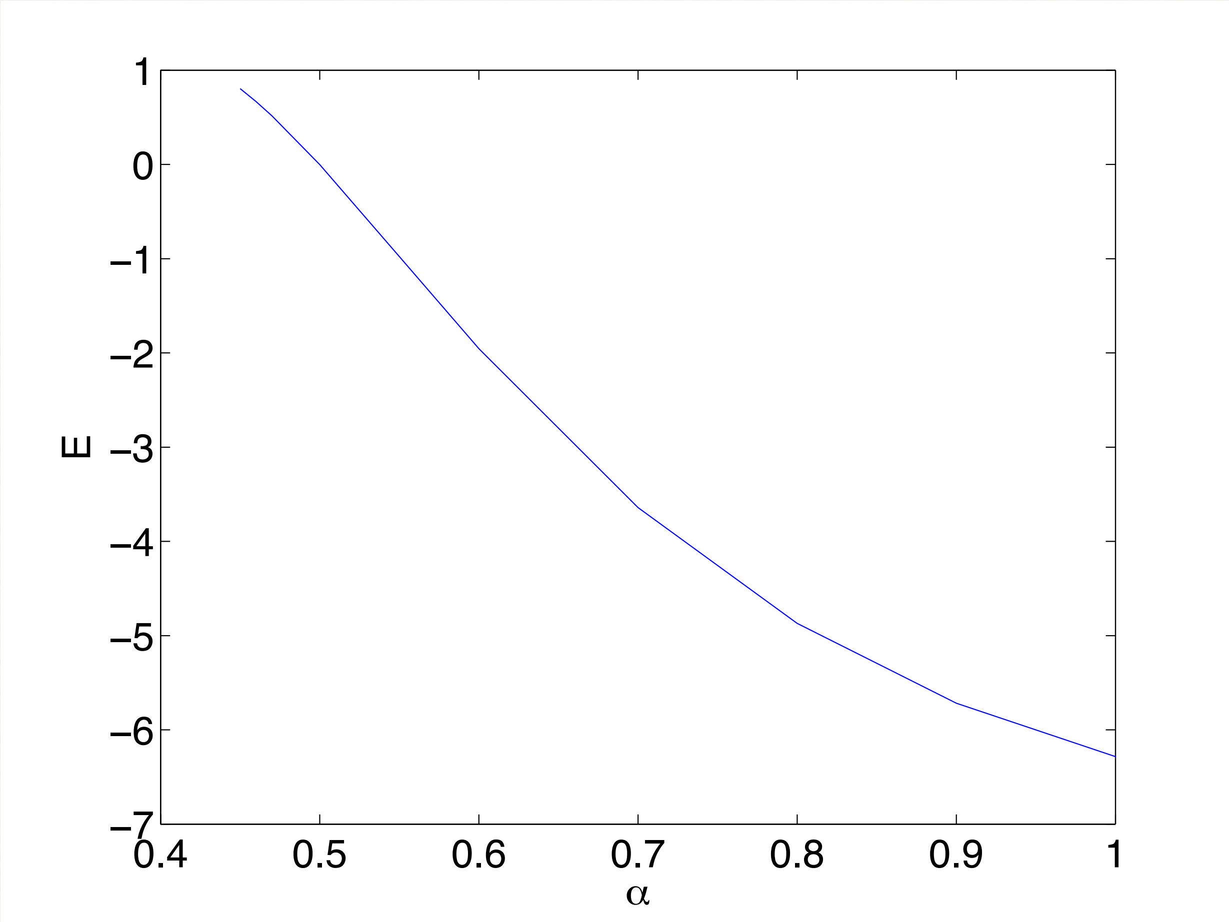
Solitary waves

- travelling wave solutions $u = Q(x - ct)$, rescalings of x and u to put $c = 1$,

$$D^\alpha Q_c + cQ_c - \frac{1}{2}Q_c^2 = 0$$

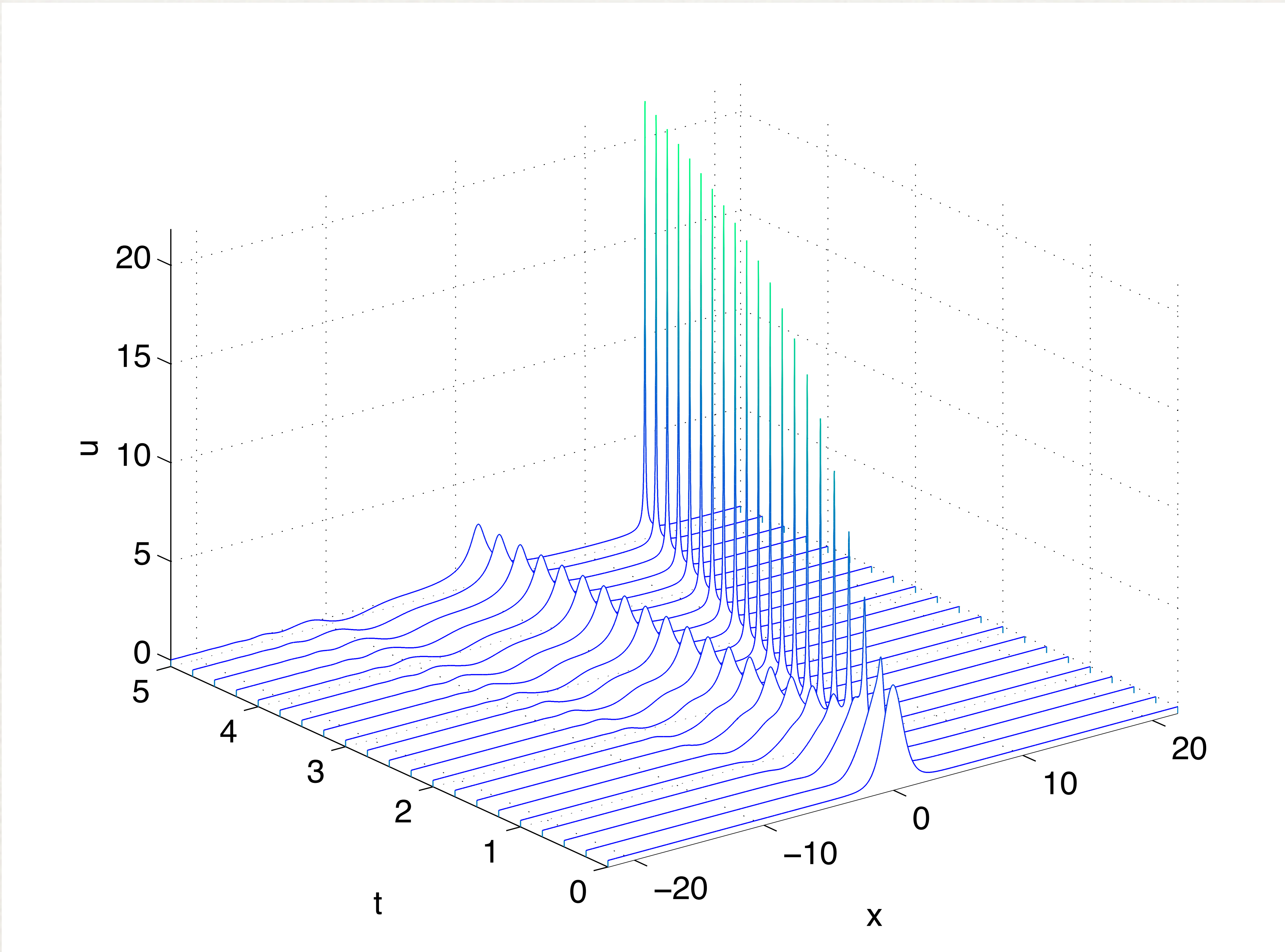
- Newton-Krylov method to construct the solutions numerically for decreasing α starting from the known solution to BO





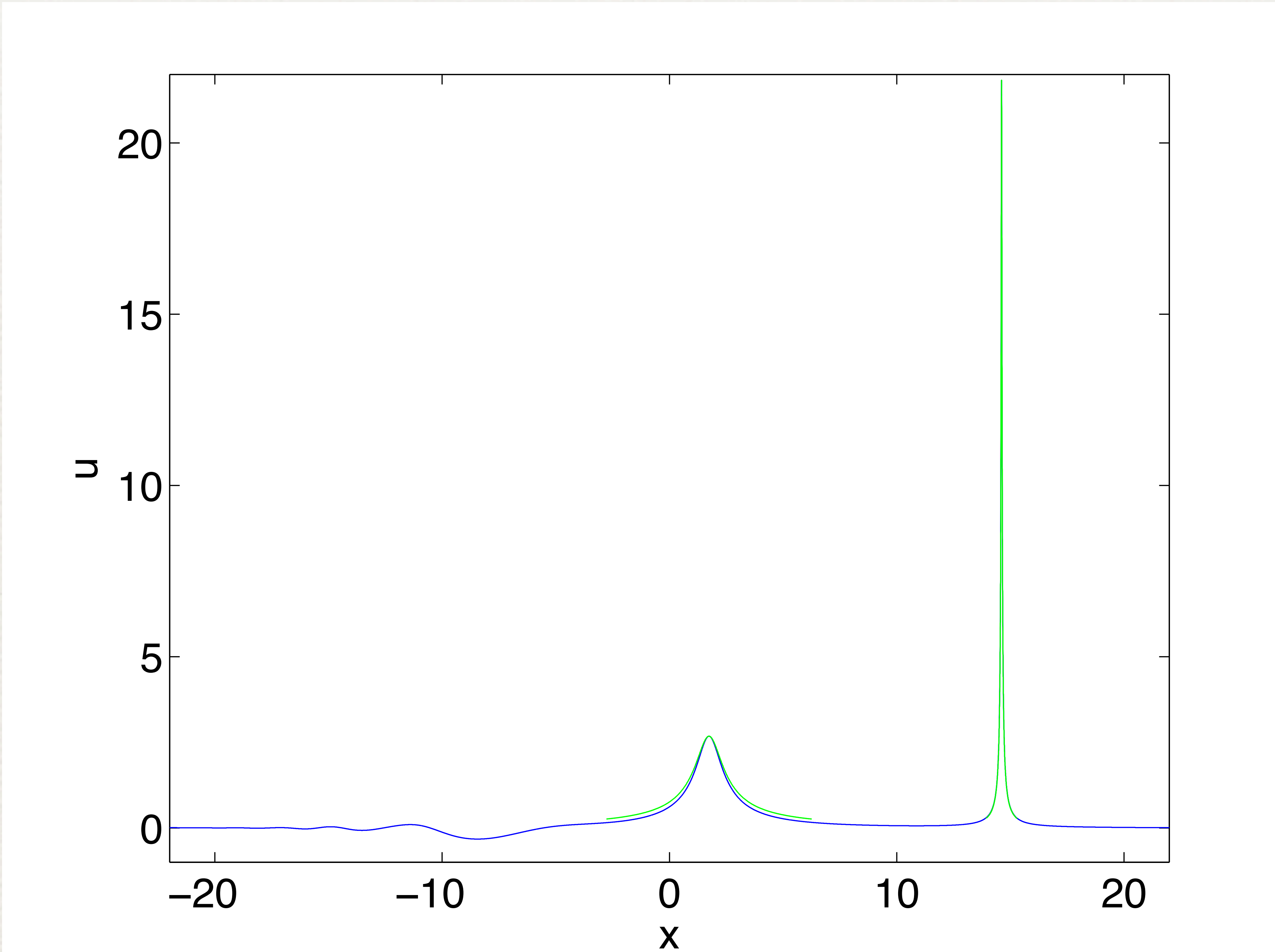
Decomposition into solitons

$$\alpha = 0.6, \quad u_0 = 5\mathrm{sech}^2 x$$



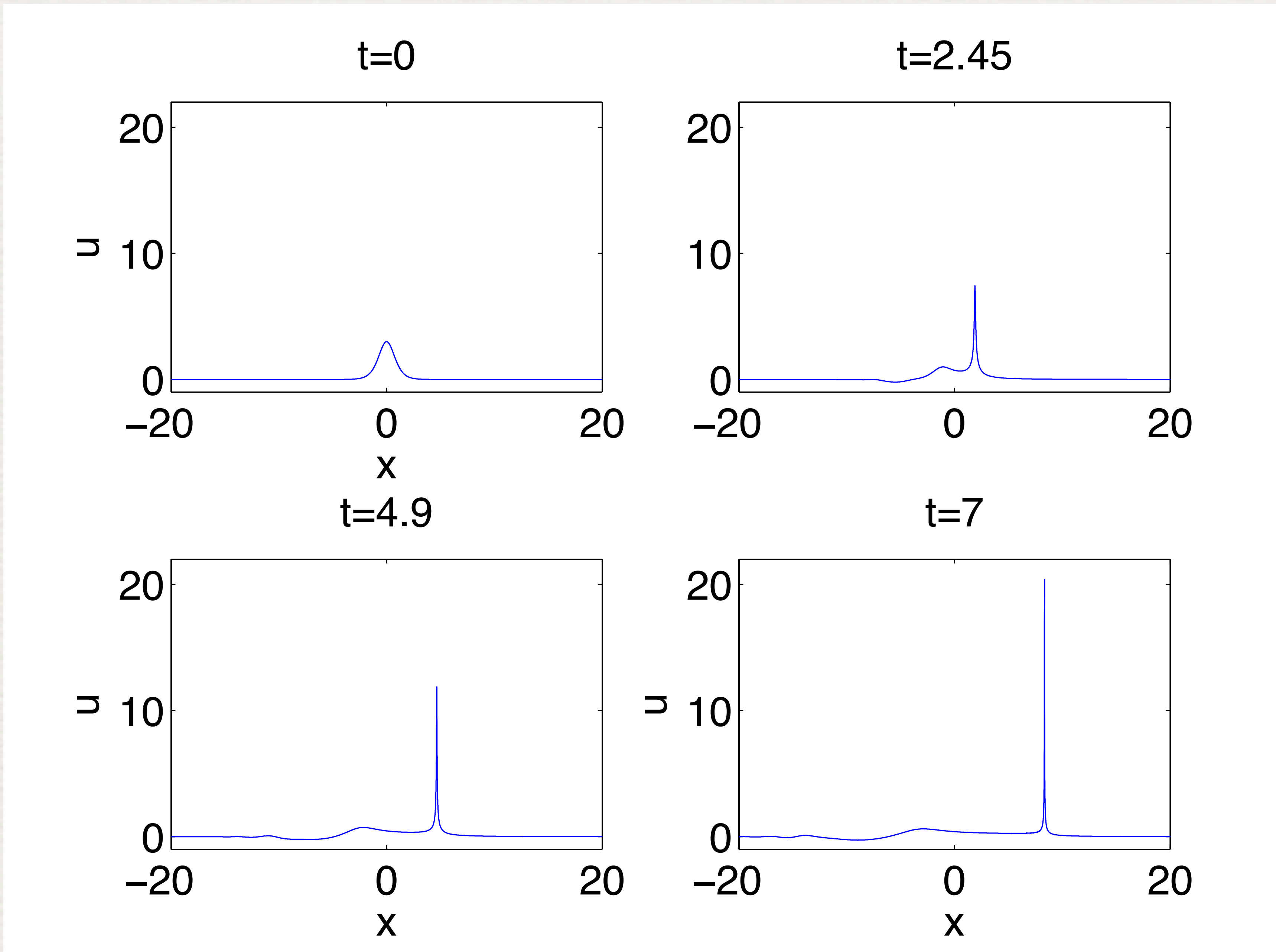
Fitted solitons

$$\alpha = 0.6, \quad u_0 = 10\text{sech}^2 x$$



Critical case

$$\alpha = 0.5, \quad u_0 = 3\text{sech}^2 x$$

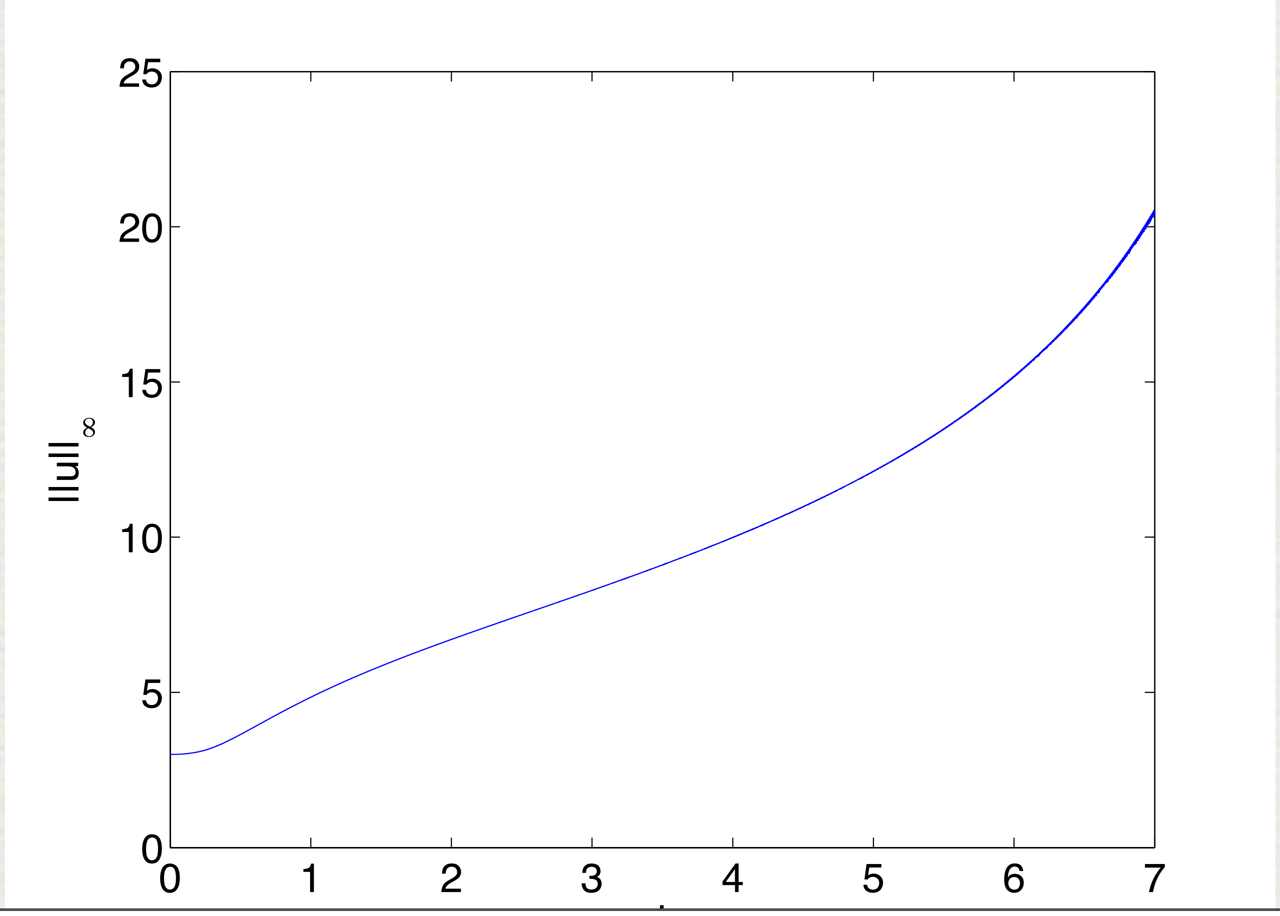


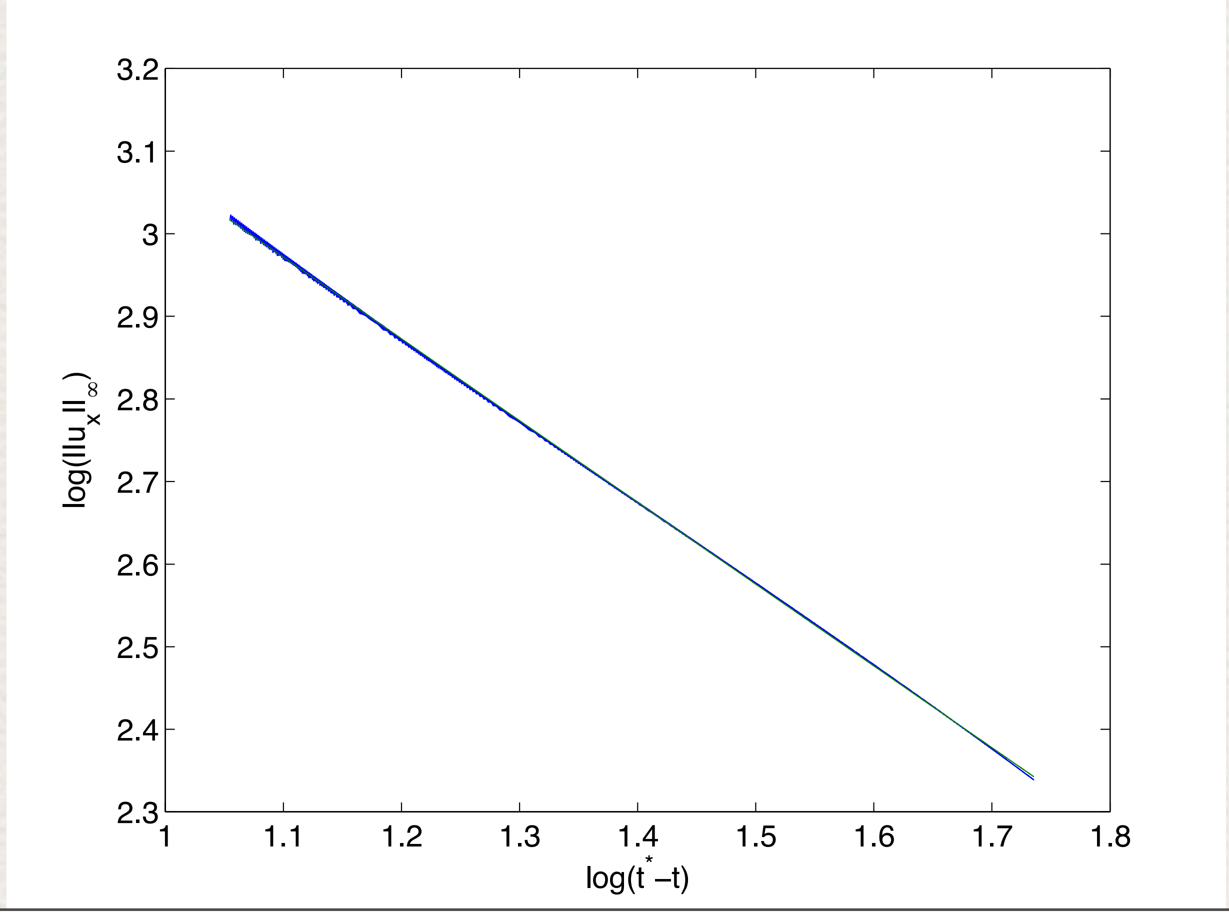
Fitting of norms of the solution

- fitting $\ln ||u||_\infty \sim \kappa_1 \ln(t^*-t)+\kappa_2$, $t^* = 9.8712$, $\kappa_1 = -0.9901$, $\kappa_2 = 4.0609$
- $L \sim \tau^{-\gamma}$ for $\tau \to \infty$:

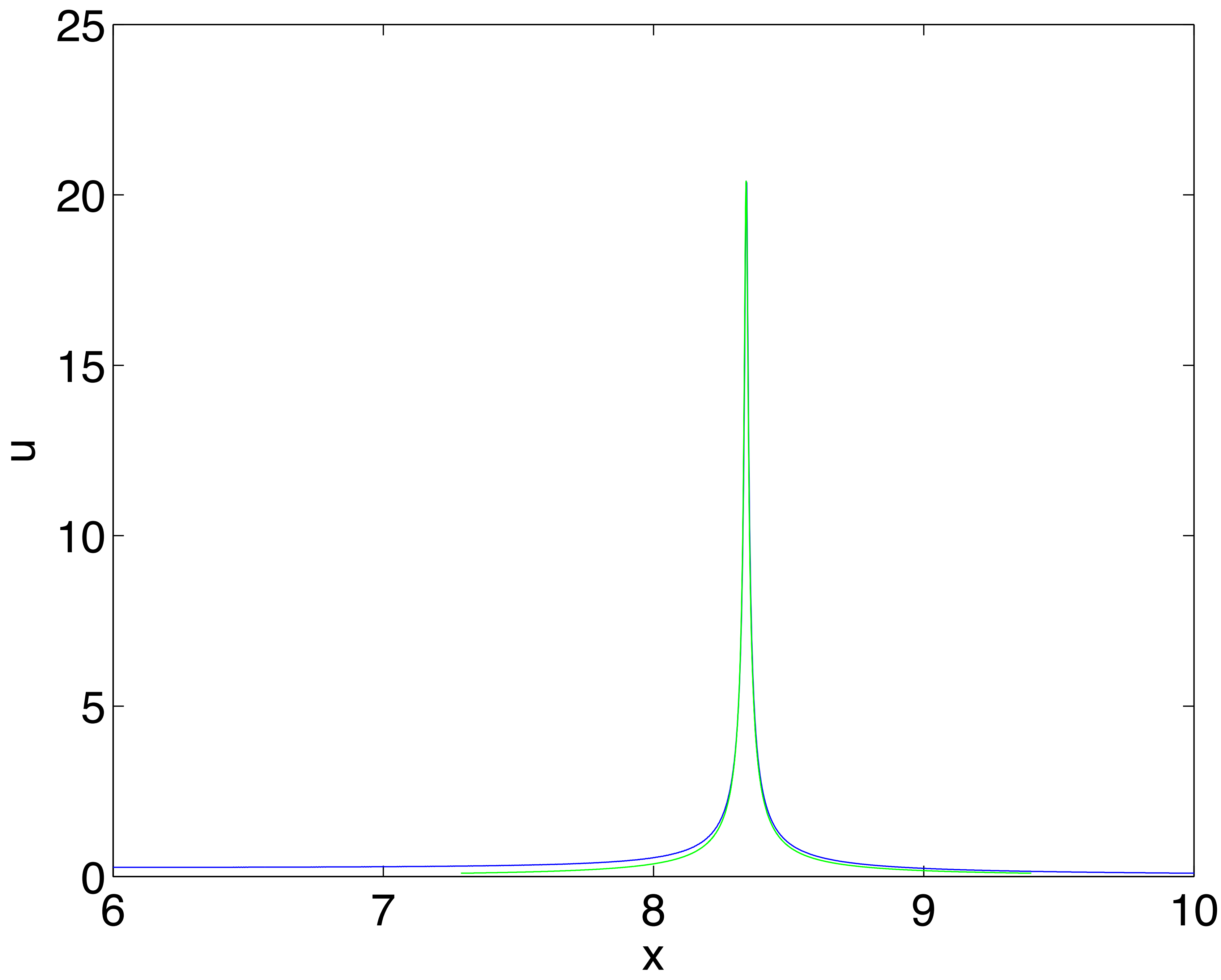
$$||u||_\infty \propto (t^* - t)^{-\frac{\alpha}{1+\alpha-1/\gamma}}$$

- as gKdV for $n = 4$, rescaled soliton





Rescaled soliton



Kadomtsev-Petviashvili equation (KP)

- Kadomtsev-Petviashvili 1970: transverse stability of KdV soliton

$$(u_t + uu_x + u_{xxx})_x + \lambda u_{yy} = 0, \quad \lambda = \pm 1$$

KPI ($\lambda = -1$): strong surface tension
KPII ($\lambda = 1$): weak surface tension

- heuristic derivation: relation of wave and advection (transport) equation, separation of left- and right-going waves,

$$(\partial_t - \partial_x)(\partial_t + \partial_x)u = 0,$$

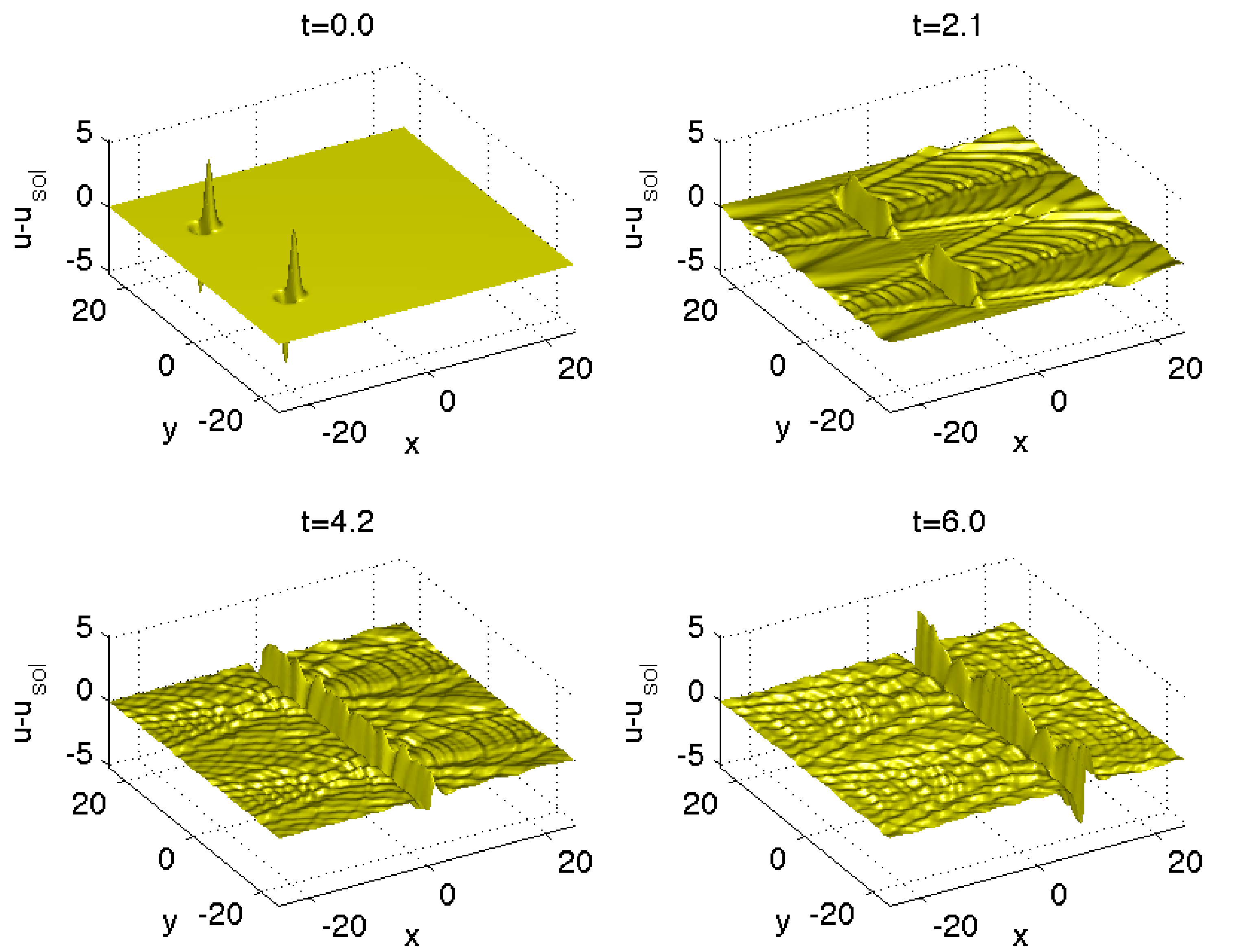
leads to $(\partial_t + \partial_x)u = 0$

Stability of exact solutions

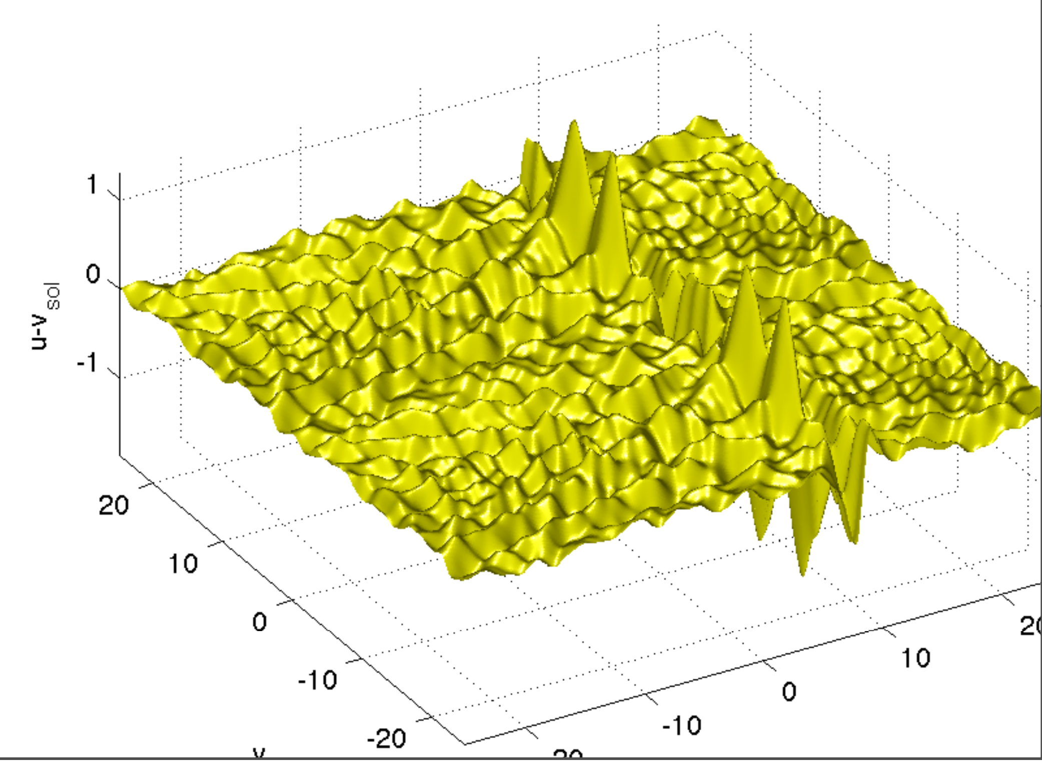
- KdV soliton lineary stable for KP II, lineary unstable for KP I (nonlinear instability by Zakharov, Rousset-Tzvetkov: orbitally stable for small amplitude)
- long time asymptotics for small initial data for KP I: lumps + radiation (Ablowitz, Fokas 1983)
- numerical study: periodic setting, exact solutions propagated with machine precision (test of precision), perturbations satisfy constraint

KP II: perturbed KdV soliton, difference u and soliton

$$u_0(x,y) = u_{sol}(x+2L_x) - 6(x+2L_x)\exp(-(x+2L_x)^2)\left(\exp(-(y+L_y\pi/2)^2) + \exp(-(y-L_y\pi/2)^2)\right)$$



- difference with fitted KdV soliton

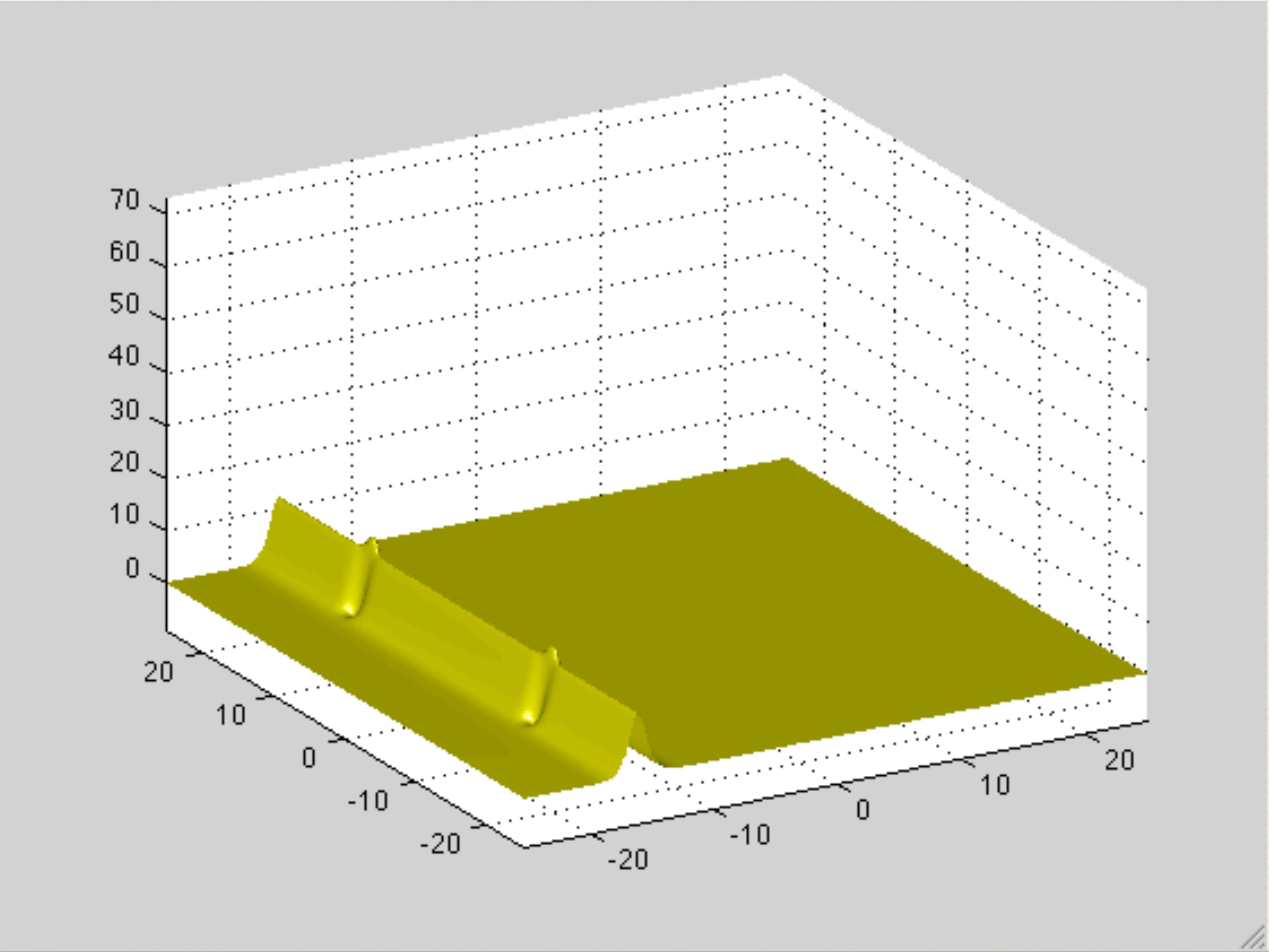


KP I: perturbed KdV soliton

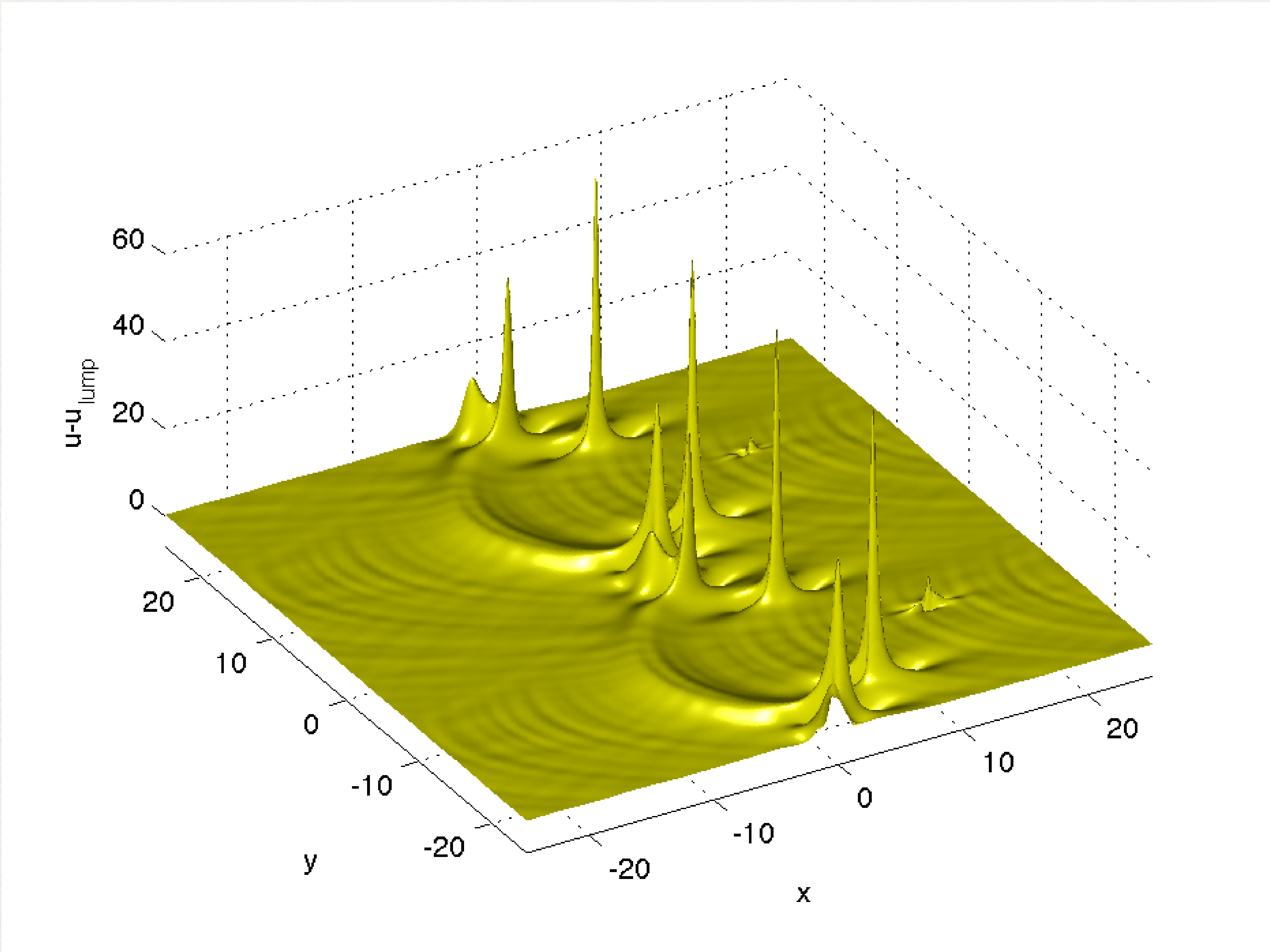
$$\begin{aligned} u_0(x,y) &= u_{sol}(x+2L_x) + 6(x+2L_x)\exp(-(x+2L_x)^2)\left(\exp(-(y+L_y\pi/2)^2)\right. \\ &\quad \left. + \exp(-(y-L_y\pi/2)^2)\right) \end{aligned}$$

KP I: perturbed KdV soliton

$$u_0(x,y) = u_{sol}(x+2L_x) + 6(x+2L_x)\exp(-(x+2L_x)^2)\left(\exp(-(y+L_y\pi/2)^2) + \exp(-(y-L_y\pi/2)^2)\right)$$



Subtract single lumps fitted at the fastest humps

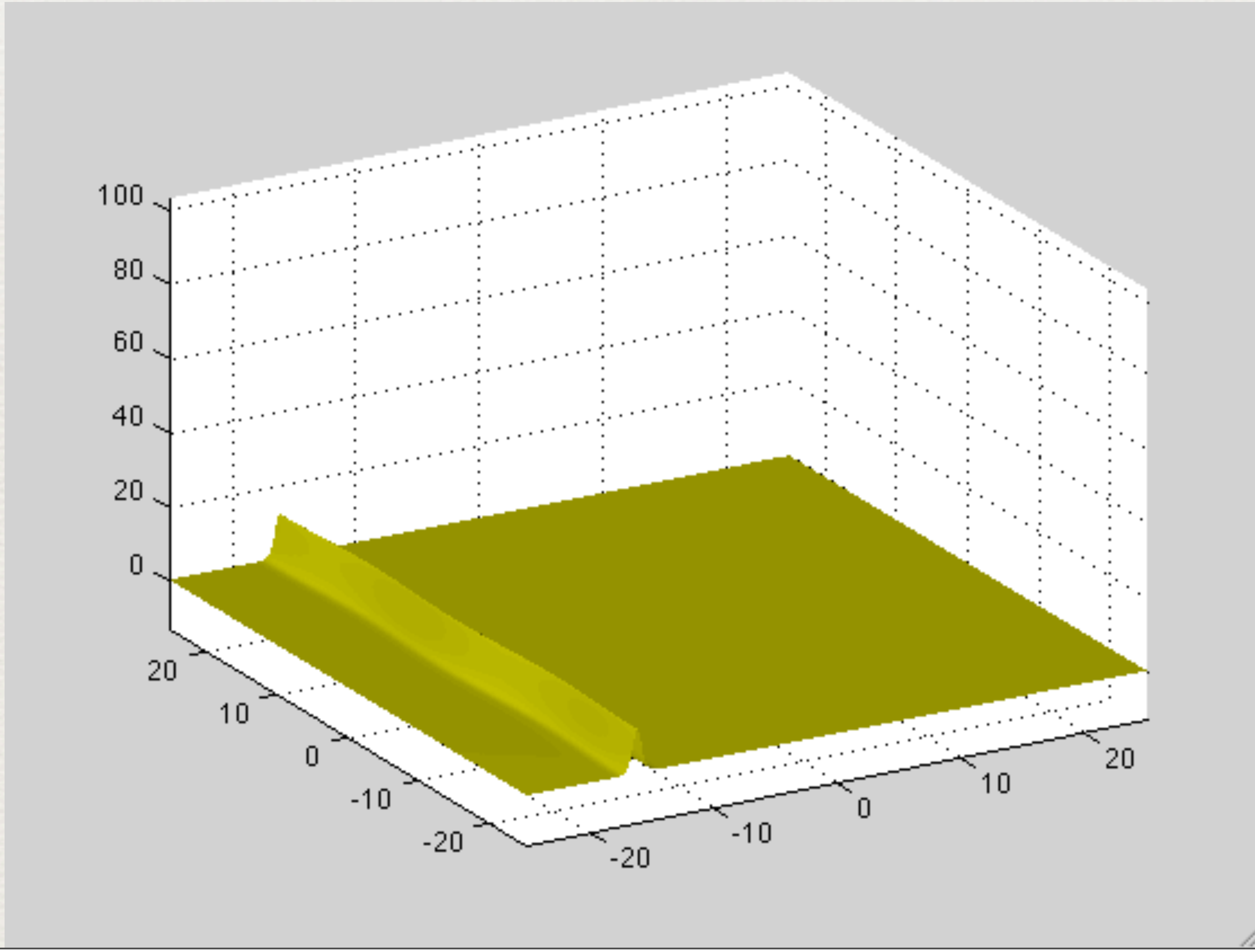


KP I: perturbed KdV soliton

$$u_0(x, y) = 12\mathrm{sech}^2(x + 0.4\cos(2y/L_y))$$

KP I: perturbed KdV soliton

$$u_0(x, y) = 12\text{sech}^2(x + 0.4\cos(2y/L_y))$$

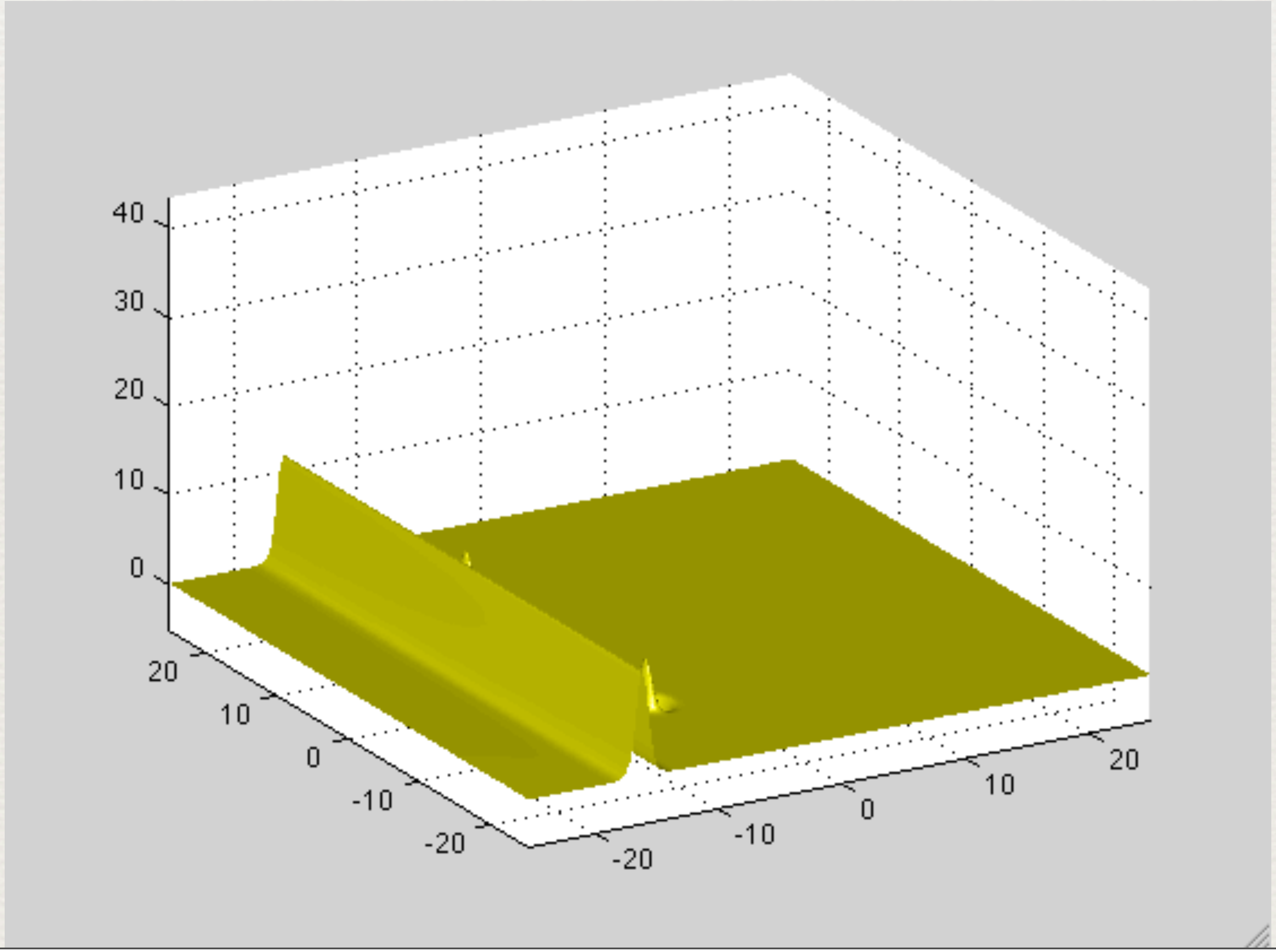


KP I: perturbed KdV soliton

$$u_0(x,y) = u_{sol}(x+2L_x) + 6(x+L_x)\exp(-(x+L_x)^2)\big(\exp(-(y+L_y\pi/2)^2) + \exp(-(y-L_y\pi/2)^2)\big)$$

KP I: perturbed KdV soliton

$$u_0(x,y) = u_{sol}(x+2L_x) + 6(x+L_x)\exp(-(x+L_x)^2)\big(\exp(-(y+L_y\pi/2)^2) + \exp(-(y-L_y\pi/2)^2)\big)$$

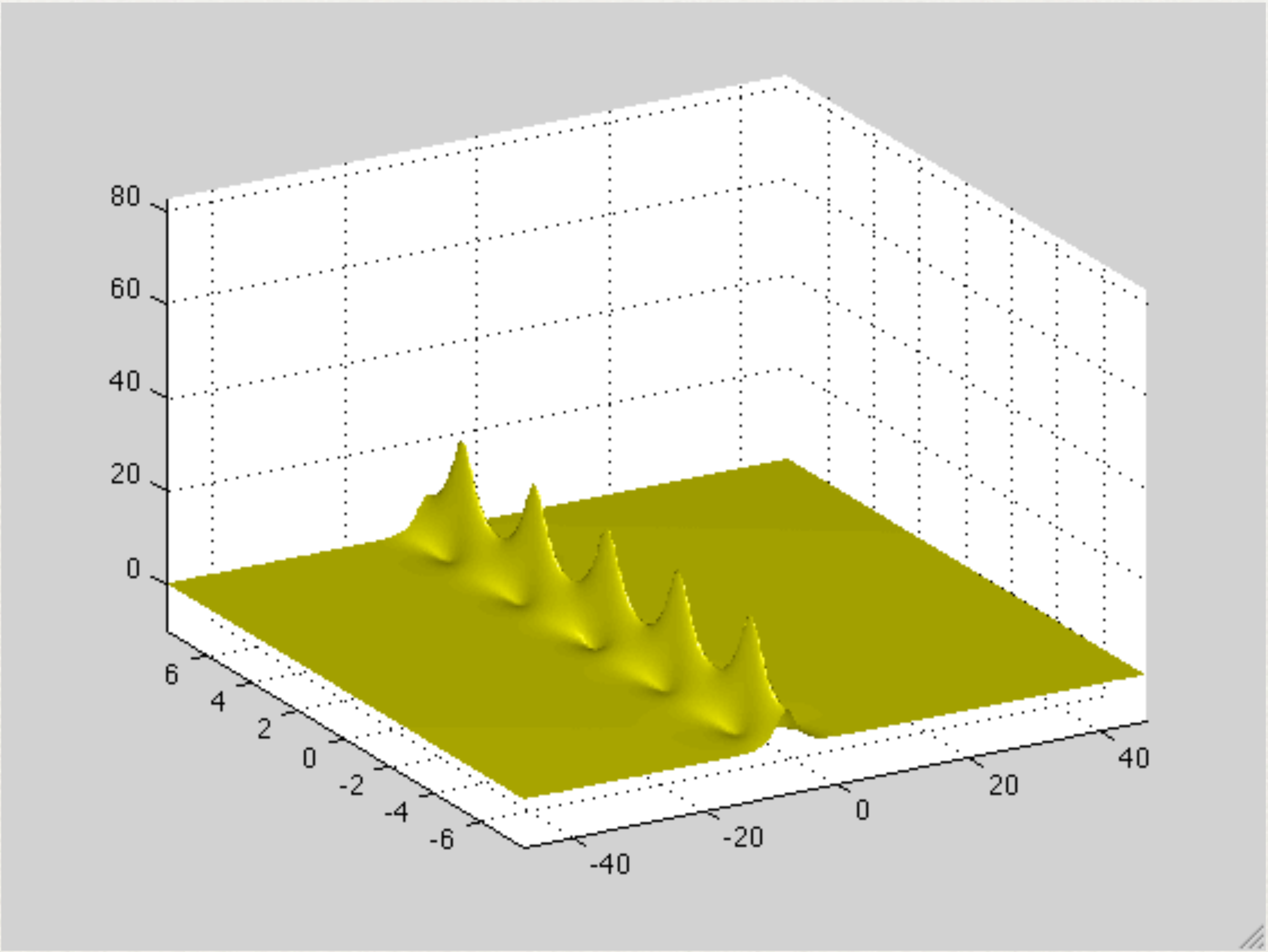


KP I: perturbed Zaitsev solution

$$u_0(x,y) = u_{zait}(x + L_x/2) + (X + L_x/2)\exp(-((X + L_x/2)^2 + Y^2))$$

KP I: perturbed Zaitsev solution

$$u_0(x,y) = u_{zait}(x + L_x/2) + (X + L_x/2)\exp(-((X + L_x/2)^2 + Y^2))$$

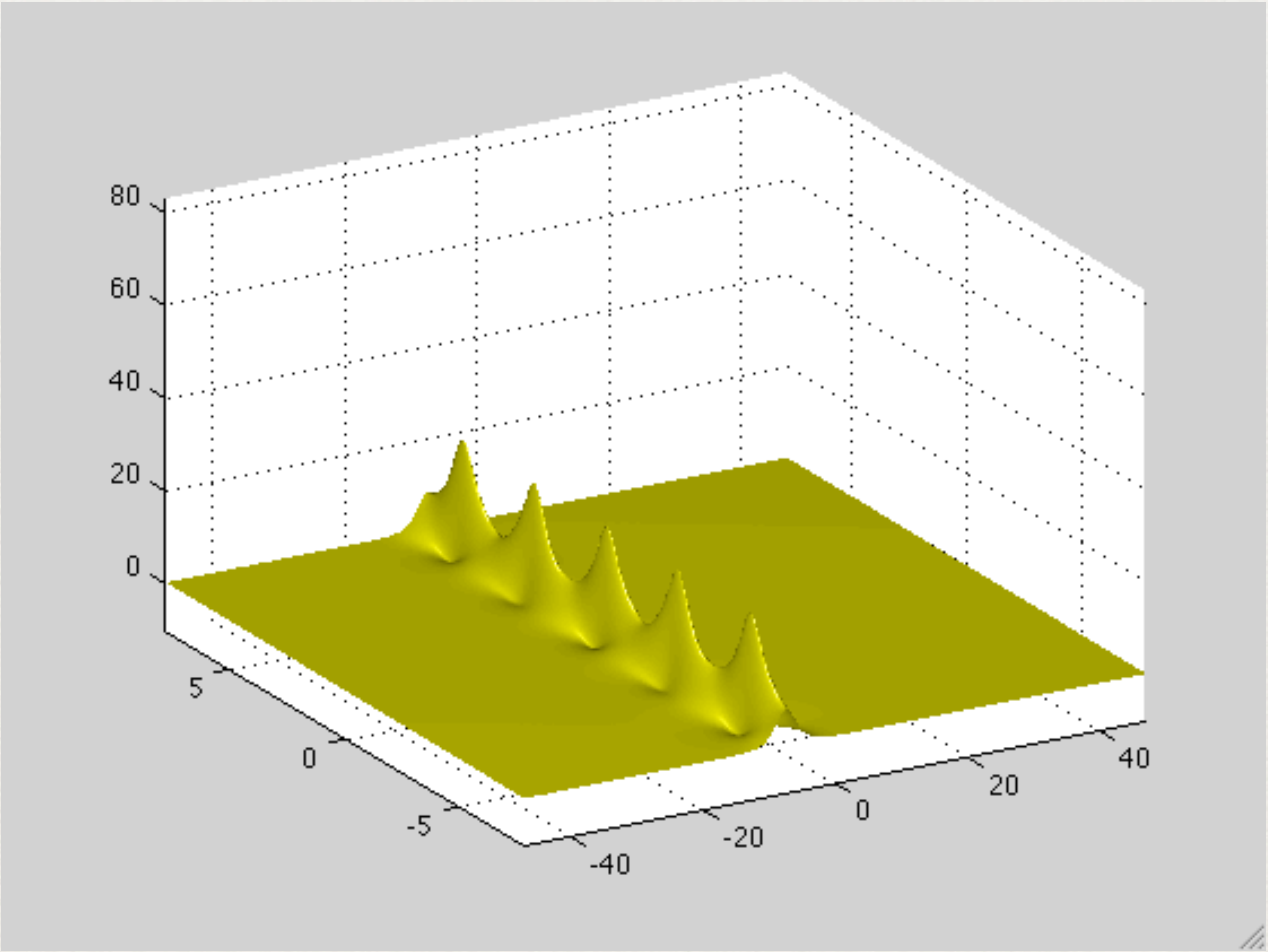


KP I: perturbed Zaitsev solution

$$u_0(x, y) = 1.1 u_{zait}(x + L_x/2)$$

KP I: perturbed Zaitsev solution

$$u_0(x, y) = 1.1 u_{zait}(x + L_x/2)$$

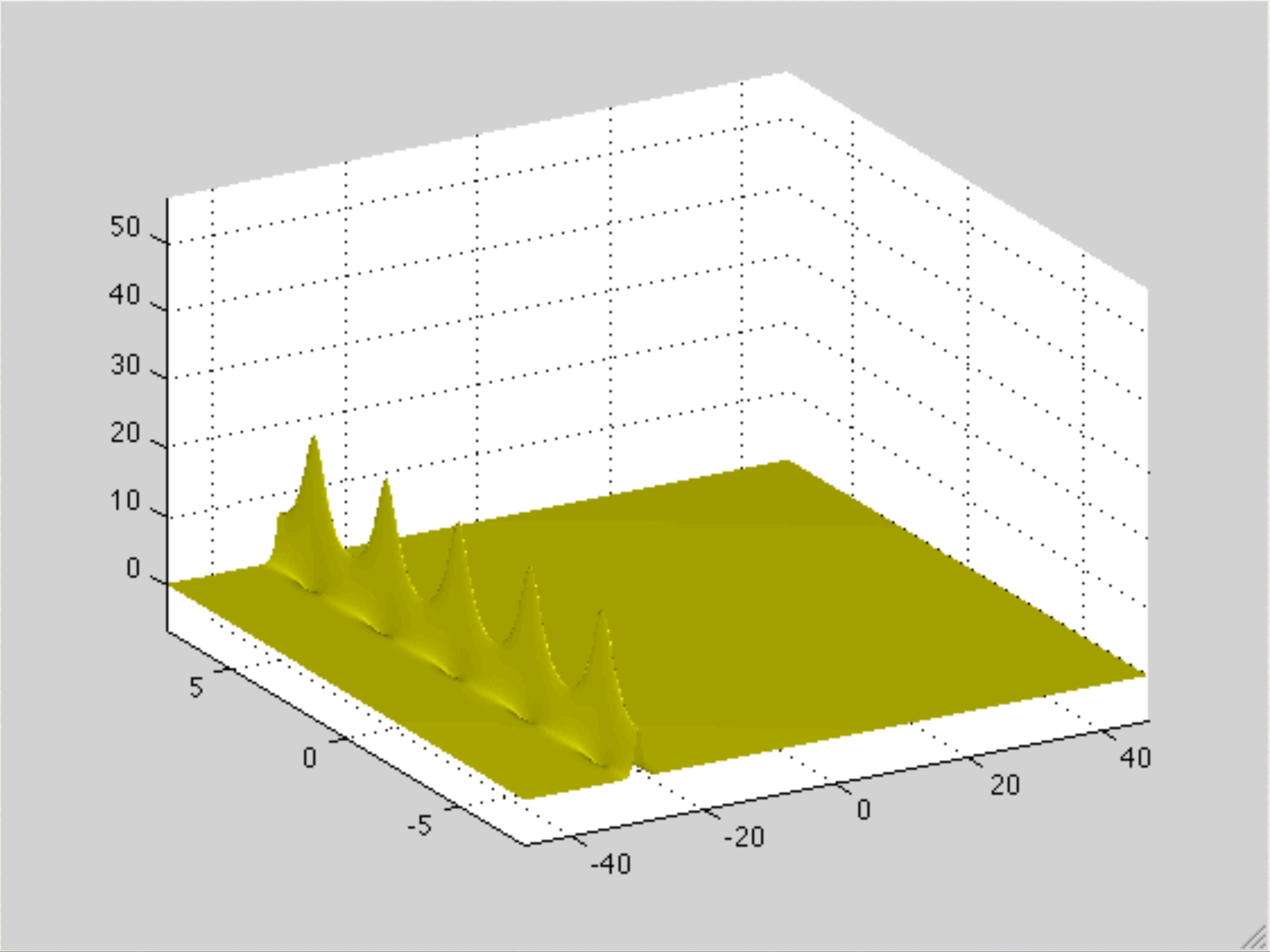


KP I: perturbed Zaitsev solution

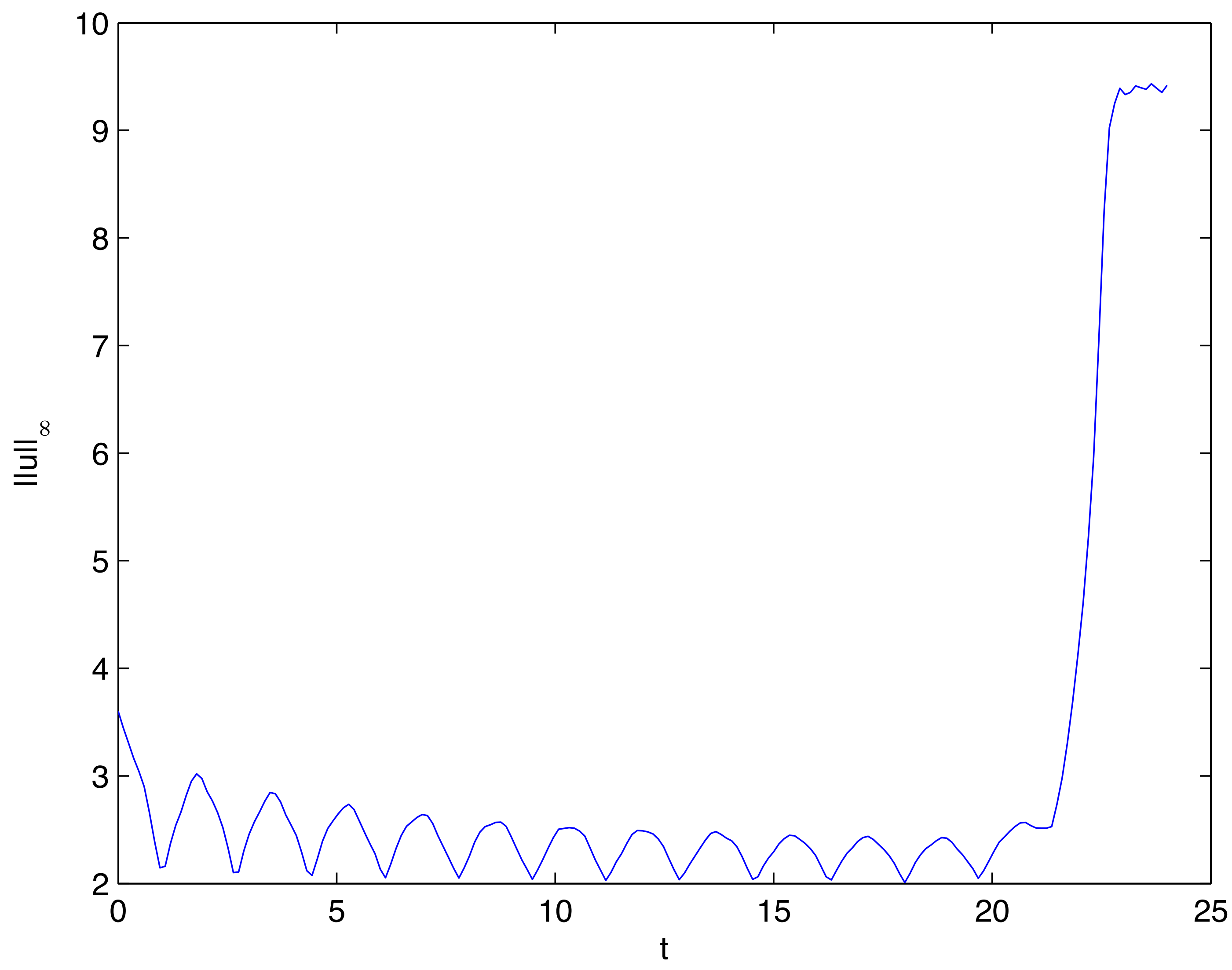
$$u_0(x, y) \quad = \quad 0.9u_{zait}(x + L_x)$$

KP I: perturbed Zaitsev solution

$$u_0(x,y) \quad = \quad 0.9u_{zait}(x+L_x)$$



KP I: perturbed Zaitsev solution



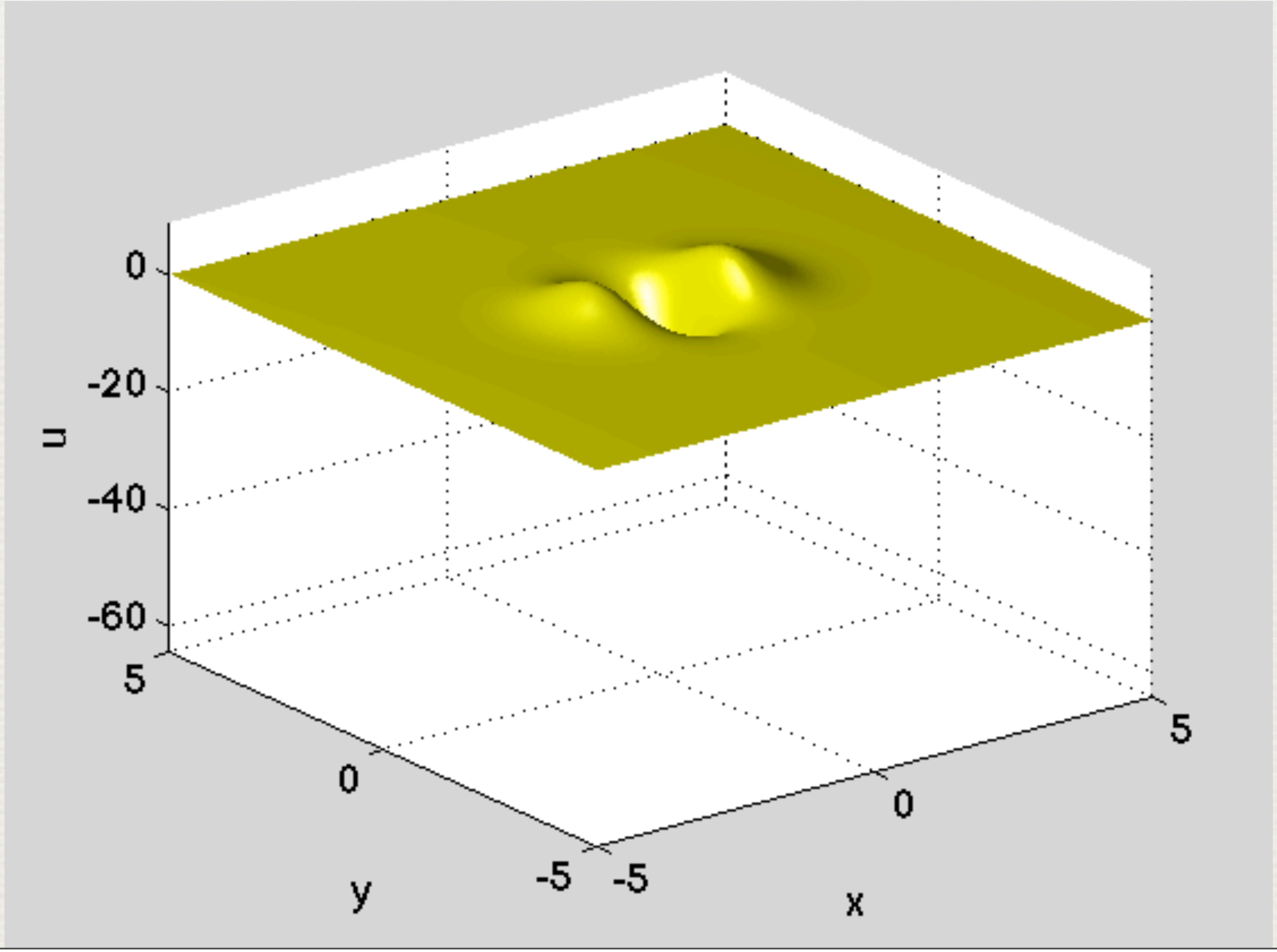
- periodic amplitude change
- oscillations around KdV soliton?
- asymptotically lumps
- KdV soliton?

Blow up in generalized KP

$$u_0(x,y) = 6\partial_{xx}\exp(-(x^2+y^2)), \quad p = 2$$

Blow up in generalized KP

$$u_0(x,y) = 6\partial_{xx}\exp(-(x^2+y^2)), \quad p=2$$



- C. Klein and J.-C. Saut, 'Numerical study of blow up and stability of solutions of generalized Kadomtsev-Petviashvili equations', J. Nonl. Sci. DOI 10.1007/s00332-012-9127-4 (2012).
- T. Grava and C. Klein, 'Numerical study of the small dispersion limit of the Korteweg-de Vries equation and asymptotic solutions' Physica D, 10.1016/j.physd.2012.04.001 (2012).
- C. Klein and C. Sparber, 'Transverse stability of periodic traveling waves in Kadomtsev-Petviashvili equations: A numerical study', (2011), *accepted for publication in Cont. Math.* arXiv:1108.3363.
- C. Klein and K. Roidot, 'Fourth order time-stepping for Kadomtsev-Petviashvili and Davey-Stewartson equations', SIAM Journal on Scientific Computing Vol. 33, No. 6, DOI: 10.1137/100816663 (2011).
- C. Klein and C. Sparber, 'Numerical simulation of generalized KP type equations with small dispersion', in 'Recent Progress in Scientific Computing', ed. by W.-B. Liu, Michael Ng and Zhong-Ci Shi, Science Press (Beijing) (2007).
- C. Klein, C. Sparber and P. Markowich , 'Numerical study of oscillatory regimes in the Kadomtsev-Petviashvili equation', J. Nonl. Sci. Vol. 17(5), 429-470 (2007).